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ON THE CLASSIFICATION OF MEASURE PRESERVING TRANSFORMATIONS
OF LEBESGUE SPACES.

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Declaration.

Section 1 has been published as a joint paper with William Parry and Peter Walters. Section 2 has been submitted for publication.

SUMMARY.

This thesis consists of three sections, each concerned with the study of the mixing properties of certain classes of measurable transformations of Lebesgue spaces. While in section 1 we consider the class of measure preserving endomorphisms of a fixed measure space (X, \mathcal{B}, m) , in sections 2 and 3 we restrict our attention to a class of piecewise monotone increasing and continuous functions of the unit interval, together with their corresponding 'natural' invariant measures.

In section 0 we give a brief description of certain degrees of mixing exhibited by endomorphisms, and two important measure theoretic invariants of a transformation, its entropy and information function. Shift spaces are also introduced and the study of shift invariant measures on these spaces is shown to correspond with the study of a large class of endomorphisms of a fixed measure space.

Section 1. A strong topology is introduced on the space of all measure preserving transformations (endomorphisms), $\text{End}(X)$, of the Lebesgue space (X, \mathcal{B}, m) . This topology distinguishes endomorphisms by measuring their 'degree of non-invertibility', indeed it renders the class of automorphisms closed and nowhere dense, and the class of zero-entropy endomorphisms nowhere dense; whereas the exact endomorphisms, regarded as the opposites of zero-entropy endomorphisms, are shown to form a dense G_δ set. The topology is related to certain conditional expectation operators on $L^2(X)$ and this lends itself to classifying $\text{End}(X)$ in terms of the information functions of these transformations. We show the set of endomorphisms which are completely characterized by their information functions form a dense set of first category, for in general, the set of endomorphisms T with information function $I_T = \infty$ a.e contains a dense G_δ set.

Similar categorization problems are considered on the class of g -measures, $\mathcal{M}_g[K]$, of a one-sided subshift of finite type. By defining an analogous topology to that defined on $\text{End}(X)$, we show that the class of measures characterized by their information functions (which by definition of g , are always bounded functions) is an open dense set. The set of functions g , with unique g -measure μ_g , is shown to contain a dense G_δ set.

Section 2. We consider the class of linear mod 1 transformations $T_{\beta\alpha}(x) = \beta x + \alpha \pmod{1}$ of the unit interval I for β and α in the ranges $1 < \beta < 2$ and $0 \leq \alpha < 1$. For $\beta \geq 2$, or $\alpha = 0$, these transformations have been shown to be weak Bernoulli (WB) with respect to a unique invariant measure, $\mu_{\beta\alpha}$, equivalent to Lebesgue measure, $[W1], [W2], [S]$. The transformations $T_{\beta\alpha}$ are known to have a unique invariant measure $\mu_{\beta\alpha}$

absolutely continuous with respect to Lebesgue measure [P], [Li,Y],[C]. By studying certain types of periodic points of the transformation $T_{\beta\alpha}$, we show that $T_{\beta\alpha}$ is WB. with respect to $\mu_{\beta\alpha}$: if and only if the support of $\mu_{\beta\alpha}$ is maximal, except for a countable set of periodic Markov transformations (Theorem 3). We give precise regions in the (β, α) -plane, in which $T_{\beta\alpha}$ is WB (Theorem 4). The transformations which are not WB are shown to have eigenvalues n^{th} -roots of unity, for some $n \in \mathbb{N}$ where $2^n \leq \beta$ (Theorem 2), and the complete set of eigenvalues are determined for each of these transformations (Theorem 6).

Section 3. Following work by Milnor and Thurston [M,T] on constructing linear models for certain continuous maps of the unit interval, I. we consider in this section similar problems for a class \mathcal{C} , of 'globally expanding' piecewise continuous and monotone functions of the unit interval. We show for $g \in \mathcal{C}$, by decomposing I into completely g -invariant irreducible sets (Theorem 1), that g is topologically semiconjugate to a uniformly piecewise linear map of I (Theorem 2), and conjugate to a piecewise linear map (Theorem 3). Further conditions are determined to ensure that $g \in \mathcal{C}$ is topologically conjugate to a uniformly piecewise linear map, thus extending a result of Parry [P2], where g was assumed to be strongly transitive. The techniques used to construct these conjugacies are similar to those used by Milnor and Thurston [M,T]. These conjugacies allow us to determine the existence of certain 'natural' g -invariant measures, $g \in \mathcal{C}$, and the measure theoretic properties of g with respect to these measures.

Preliminary Definitions and Ergodic Theory Results.

Throughout, all measure spaces will be Lebesgue spaces and all transformations of these spaces will be measurable.

Let $T : X \rightarrow X$ denote a measurable transformation of the Lebesgue space (X, \mathcal{B}, m) .

T is measure preserving if $m(B) = m(T^{-1}B)$ for all $B \in \mathcal{B}$.

Definition 1a. If $(X_1, \mathcal{B}_1, m_1)$ and $(X_2, \mathcal{B}_2, m_2)$ are Lebesgue spaces and $T_1 : X_1 \rightarrow X_1$, and $T_2 : X_2 \rightarrow X_2$ are measure preserving transformations then T_1 is isomorphic to T_2 if there exists

(i) $\underline{X}_i \in \mathcal{B}_i$, with $m_i(\underline{X}_i) = 1$ and $T_i \underline{X}_i \subset \underline{X}_i$, $i = 1, 2$,
and (ii) an invertible measure preserving transformation $\phi : \underline{X}_1 \rightarrow \underline{X}_2$
such that $\phi T_1(x) = T_2 \phi(x)$, $x \in \underline{X}_1$.

Write $T_1 \approx T_2$.

Definition 1b. Let T_1 and T_2 be measurable transformations of the

Borel spaces (X_1, \mathcal{B}_1) , (X_2, \mathcal{B}_2) . We say

- (i) T_1 is topologically conjugate to T_2 if there is a homeomorphism $\phi : X_1 \rightarrow X_2$ such that $T_2 = \phi T_1 \phi^{-1}$.
(ii) T_1 is topologically semiconjugate to T_2 if there is a continuous map $\psi : X_1 \rightarrow X_2$ such that $\psi T_1 = T_2 \psi$, (T_2 is conjugate to a factor of T_1).

Generally a measure preserving transformation $T : X \rightarrow X$ will be called an endomorphism, or an automorphism if it is invertible.

Definition 2.a) A partition of X is a family $\alpha = \{A_i\}_{i \in I}$ of a.e. disjoint sets where $A_i \in \mathcal{B}$ and $\bigcup_{i \in I} A_i = X$ a.e.

b) The sets A_i of the partition α are called atoms of α .

c) $\hat{\alpha}$ is the smallest σ -algebra containing α .

d) If α and γ are partitions their refinement $\alpha \vee \gamma$ is the partition $\{A_i \cap B_j\}_{i \in I, j \in J}$, where $\alpha = \{A_i\}_{i \in I}$, $\gamma = \{B_j\}_{j \in J}$.

e) $\bigvee_{i=0}^n T^{-i} \alpha = \alpha \vee T^{-1} \alpha \vee \dots \vee T^{-n} \alpha$; $\alpha^- = \bigvee_{i=1}^{\infty} T^{-i} \alpha$.

f) $\alpha_n \uparrow \alpha$ if $\alpha_{n-1} \vee \alpha_n = \alpha_n$ and $\bigvee_0^\infty \alpha_n = \alpha$.

g) α is a generator of the endomorphism T , if $(\alpha \vee \alpha^-)^\wedge = \mathcal{B}$.

α is a generator of the automorphism T , if $(\bigvee_{-\infty}^\infty T^{-n} \alpha)^\wedge = \mathcal{B}$.

h) \mathcal{E} is the point partition, that is, $\hat{\mathcal{E}} = \mathcal{B}$.

i) η is the trivial σ -algebra $\{X, \emptyset\}$.

Conditional Expectation, Information and Entropy. Details of the following can be found in [PQ].

Suppose \mathcal{C} is a sub- σ -algebra of \mathcal{B} and $f \in L_1(X, \mathcal{B}, m)$ then there is an essentially unique \mathcal{C} -measurable function $E(f/\mathcal{C})$ such that

$$\int_C E(f/\mathcal{C}) \, dm = \int_C f \, dm \quad \text{for all } C \in \mathcal{C}.$$

$E(f/\mathcal{C})$ is the conditional expectation of f given \mathcal{C} .

For $B \in \mathcal{B}$ the conditional probability of B given \mathcal{C} , denoted by $m(B/\mathcal{C})$, is defined by $m(B/\mathcal{C}) = E(\chi_B/\mathcal{C})$ where χ_B is the characteristic function of B .

If $\alpha = \{A_i\}_{i \in \mathbb{N}}$ is a countable partition of X , then

$$E(f/\alpha) = \sum_{i=0}^{\infty} \chi_{A_i} \left(\int_{A_i} f \, dm / m(A_i) \right) \quad \text{and}$$

$$E(\alpha/T^{-1}\mathcal{B}) = \sum_{i=0}^{\infty} \chi_{A_i} m(A_i/T^{-1}\mathcal{B}).$$

These concepts can be generalized to arbitrary measurable partitions. In particular, if we consider the partitions \mathcal{E} and $T^{-1}\mathcal{E}$, then on each fibre $\{T^{-1}x\}$ of $T^{-1}\mathcal{E}$ there exists a canonical measure $m(\cdot/T^{-1}x)$. If $\{T^{-1}x\}$ is countable then this is an atomic measure but if $\{T^{-1}x\}$ is not countable then it could be continuous.

We define $E(\mathcal{E}/T^{-1}\mathcal{E})(x) = m(x/T^{-1}x) = \lim_{n \rightarrow \infty} E(\beta_n/T^{-1}\mathcal{B})$,

where β_n are finite partitions and $\beta_n \uparrow \mathcal{E}$ and $x \in X$.

We now introduce two important isomorphism invariants of an endomorphism, the information function $I(\mathcal{B}/T^{-1}\mathcal{B})$ and the entropy $h(T)$.

Definition 3. a) If α is a countable partition of X then the information of α with respect to \mathcal{C} , written $I(\alpha/\mathcal{C})$, is given by

$$I(\alpha/\mathcal{C}) = -\sum_{i=1}^{\infty} \chi_{A_i} \log m(A_i/\mathcal{C}).$$

b) The information function $I(\mathcal{B}/T^{-1}\mathcal{B})$ is given by

$$I(\mathcal{B}/T^{-1}\mathcal{B}) = -\log E(\mathcal{B}/T^{-1}\mathcal{B}).$$

Definition 4. a) Let α be a countable partition of X , the conditional entropy of α with respect to \mathcal{C} $H(\alpha/\mathcal{C})$ is given by

$$H(\alpha/\mathcal{C}) = \int I(\alpha/\mathcal{C}) \, dm.$$

b) The entropy of the partition α $h(T, \alpha)$ is given by

$$h(T, \alpha) = H(\alpha/\hat{\alpha}^-).$$

c) The entropy of the transformation T $h_m(T)$ is given by

$$h_m(T) = \sup h(T, \alpha) \leq \infty,$$

where the supremum is taken over all finite partitions of X .

Theorem (Kolmogorov and Sinai). If α is a finite generator of T then

$$h_m(T) = h(T, \alpha).$$

Maximal Measures. Consider the measurable transformation T acting on the Borel space (X, \mathcal{B}) .

Let $\mathcal{M}_T = \{\mu : \mu \text{ is a } T\text{-invariant probability measure on } X\}$ and

let $h(T) = \sup_{\mu \in \mathcal{M}_T} h_{\mu}(T) \leq \infty$.

Definition 5. μ is a maximal measure for T if $h_{\mu}(T) = h(T)$.

It is easy to show that maximal measures, if they exist, are preserved under topological conjugacies.

Mixing. In section 1 we partially classify the elements of $\text{End}(X)$ in terms of their randomness, that is, the degree to which they mix the points of the space X . The following properties of measure preserving transformations qualitatively describe this randomness and they are all isomorphism invariants, [W].

Definition 6. Let T be a measure preserving transformation of the

Lebesgue space (X, \mathcal{B}, m) , then

- (i) T is aperiodic if $m(\{x \in X: T^n x = x, \text{ for some } n > 0\}) = 0$.
- (ii) T is ergodic if $T^{-1}B = B$ for some $B \in \mathcal{B}$ then $m(B) = 0$ or $m(B) = 1$.

- (iii) T is weak mixing if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} |m(A \cap T^{-n}B) - m(A)m(B)| = 0, \text{ for all } A, B \in \mathcal{B}.$$

- (iv) T is strong mixing if

$$\lim_{N \rightarrow \infty} m(T^{-N}A \cap B) = m(A)m(B) \text{ for all } A, B \in \mathcal{B}.$$

- (v) T is exact if $\bigcap_n T^{-n}\mathcal{B} = \eta$.

- (vi) T is Markov if it has a countable generator α such that

$$m(A_{i_0} \cap T^{-1}A_{i_1} \cap \dots \cap T^{-n}A_{i_n}) = \lambda_{i_0} p_{i_0 i_1} \dots p_{i_{n-1} i_n},$$

where p_{ij} are entries in a square stochastic matrix P with $\underline{\lambda} = (\lambda_i)$ a strictly positive left invariant vector of P , $m(A_{i_0}) = \lambda_{i_0}$, $A_{i_j} \in \alpha$.

- (vii) T is Bernoulli if it has a countable generator α such

$$\text{that } m(A_{i_0} \cap T^{-1}A_{i_1} \cap \dots \cap T^{-n}A_{i_n}) = m(A_{i_0}) \dots m(A_{i_n}),$$

$A_{i_j} \in \alpha$. We sometimes say (T, α) is a Bernoulli

process.

For the following results and other elementary results see [W].

Theorem. On a non-atomic space,

T Bernoulli \Rightarrow T aperiodic and Markov \Rightarrow T exact \Rightarrow

T strong mixing \Rightarrow T weak mixing \Rightarrow T ergodic \Rightarrow T aperiodic.

Ornstein has shown that entropy is a complete invariant for Bernoulli automorphisms [O]. In fact, he shows that entropy is a complete invariant for the class of weak Bernoulli automorphisms.

Definition 7. [O] (i) Let α and γ be two countable partitions of I , we

say α and γ are ε -independent if

$$D(\alpha, \gamma) = \sum_{\substack{A_i \in \alpha, \\ C_i \in \gamma}} |m(A_i \cap C_i) - m(A_i)m(C_i)| < \varepsilon.$$

(ii) If T is an automorphism, with generator α , such that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ so that

$$D\left(\bigvee_{n=0}^k T^{-n}\alpha, \bigvee_{n=k+N}^{2k+N} T^{-n}\alpha\right) < \varepsilon, \text{ for all } k \geq 0,$$

then we say T is a weak Bernoulli automorphism.

(iii) We say the endomorphism T is weak Bernoulli provided its natural extension \tilde{T} [R2] is a weak Bernoulli automorphism.

It has been shown [P, W] that neither entropy nor even the information function is a complete isomorphism invariant for $\bigwedge_{\mathbb{N}}^{\text{WB}}$ endomorphisms.

Shift spaces. Let Y be a compact metric space, and let

$$X = \prod_{i=0}^{\infty} Y_i \text{ and } X' = \prod_{i=-\infty}^{\infty} Y_i, \text{ where } Y_i = Y, i \in \mathbb{N}.$$

$X(X')$ is the space of all one(two)-sided sequences $\underline{x} = x_0 x_1 \dots$

($\underline{x}' = \dots x_{-1}' x_0' x_1' \dots$) $x_i \in Y$.

With respect to the product topology, the shift map $T(T')$ defined by

$T(\underline{x}) = \underline{y}$, where $x_i = y_{i-1}$, $i \geq 0$, ($T'(\underline{x}') = \underline{y}'$, where $x_i' = y_{i-1}'$, $i \in \mathbb{Z}$),

is a continuous map on $X(X')$.

We call the sets $[i_0 i_1 \dots i_n] = \{\underline{x} : x_0 = i_0, x_1 = i_1, \dots, x_n = i_n\}$,
 $([i_0 i_1 \dots i_n]' = \{\underline{x}' : x'_0 = i_0, \dots, x'_n = i_n\})$, cylinder sets of $X, (X')$.
 Let $\mathcal{B} (\mathcal{B}')$ be the smallest $T(T')$ -invariant σ -algebra containing
 all cylinder sets.

It is easy to see that $\{[i_0]\}_{i_0 \in Y}$ and $\{[i_0]'\}_{i_0 \in Y}$ are generators
 for T and T' respectively.

We call (X, \mathcal{B}) a one-sided shift space and (X', \mathcal{B}') a two-sided
shift space.

If Λ is a closed T -invariant subset of X then we call $(\Lambda, \mathcal{B}|_{\Lambda})$
 a one-sided subshift.

Most important are the cases when Y is a finite set, or the unit interval.

Definition 8. Let $\bigwedge_{i=0}^{\infty} Y_i$ be a subshift, where $Y_i = Y$, $i \in \mathbb{N}$ and Y is

a finite set with m elements.

We call Λ a one-sided subshift of finite type (s.s.f.t.)

if there exists an $m \times m$ matrix A with 0,1 entries only,

such that $\underline{x} \in \Lambda$ if and only if $A_{x_n x_{n+1}} = 1$ for all $n \in \mathbb{N}$.

If A^M has all entries positive for some $M \in \mathbb{N}$ then Λ is
 called a one-sided mixing (or aperiodic) subshift of
finite type.

We shall be interested in subshifts of finite type in particular,
 since $(\Lambda, \mathcal{B}, T)$ is a subshift of finite type (or intrinsically Markov)
 if and only if T is locally a homeomorphism [P2]. Subshifts of finite
 type play a fundamental role in solving many topological and measure
 theoretic problems in Ergodic Theory.

Definition 9. Let μ be a probability measure on the shift space

(X, \mathcal{B}, T) defined above, then we say

- (i) μ is Markov, or (X, μ) is a Markov shift, if T is a
 Markov transformation with respect to the partition

$$\{[i_0]\}_{i_0 \in Y}$$

- (ii) μ is Bernoulli, or (X, μ) is a Bernoulli shift, if T is a Bernoulli transformation with respect to the generator $\{[i_0]\}_{i_0 \in Y}$.

Definition 10. [0] Let $X = \prod_0^{+\infty} [0, 1]$. Let Σ be the smallest shift

invariant σ -algebra generated by the cylinders

$[c_1 c_2 \dots c_n]$, where $c_i \in \mathcal{B}$, the Borel σ -algebra of $[0, 1]$.

Let μ be a shift invariant probability measure defined on X by, $\mu[c_1 c_2 \dots c_n] = \mu[c_1] \mu[c_2] \dots \mu[c_n]$, then

(X, Σ, μ) is called the Generalized Bernoulli Shift.

The correspondence between $\text{End}(X)$ and shift invariant measures.

Let $Y = \prod_0^{\infty} \mathbb{N}$ and \mathcal{B}_Y be the Borel σ -algebra generated by cylinder sets.

Let T be an endomorphism of the Lebesgue space (X, \mathcal{B}, m) with finite entropy, or with countable inverse images of points, then T has a finite or countable generator $\alpha, [\mathbb{R}^3]$.

Let $\alpha = (A_i)_{i \in \mathbb{N}}$.

Define a map $\phi: (X, \mathcal{B}, m) \rightarrow (Y, \mathcal{B}_Y)$, by

$$\phi(x) = \underline{y}, \text{ if } x \in T^{-i} A_{y_i} \text{ for all } i \in \mathbb{N} \text{ where } \underline{y} = y_0 y_1 \dots$$

ϕ is well defined a.e. on X .

Define a measure μ on Y by $\mu(C) = m(\phi^{-1}C)$ for $C \in \mathcal{B}_Y$.

It is easy to check that μ is a shift invariant measure and that the shift S acting on (Y, \mathcal{B}_Y, μ) is measure theoretically isomorphic to

T acting on (X, \mathcal{B}, m) , (by the isomorphism ϕ).

Therefore the classification of endomorphisms with countable generators on (X, \mathcal{B}, m) is analogous to the classification of shift invariant measures on (Y, \mathcal{B}_Y) .

Throughout the thesis we shall refer to this analogy either explicitly or implicitly.

SECTION 1.

LARGE SETS OF ENDOMORPHISMS AND OF g -MEASURES

These notes are the outcome of an unfulfilled attempt to prove that, in a reasonable sense, most endomorphisms of a Lebesgue space (X, \mathcal{B}, m) can be classified by a countable number of invariants. We attempted to do this despite Feldman's negative warnings in [F] concerning automorphisms; and in fact the invariants we had in mind will not classify. The complete metric topology we adopt for the space $E(X)$ of endomorphisms is stronger than the usual weak operator topology since we considered that the latter to be inappropriate. In fact the set $A(X)$ of automorphisms of X is dense in $E(X)$ in the weak topology. This is one basic reason for adopting what we call the strong adjoint topology. In this topology $A(X)$ is a closed nowhere dense subset of $E(X)$. Another reason for adopting the strong adjoint topology is that certain conditional expectation operators are directly related to this topology.

With the strong adjoint topology the set of exact endomorphisms is a dense G_δ in $E(X)$ whereas with the weak topology even the strong-mixing endomorphisms are a set of first category. We shall also show that the set of exact Markov endomorphisms and even the 'irregular' exact Markov endomorphisms are dense in $E(X)$ with respect to the strong adjoint topology. An endomorphism T is irregular if its information function $I(\mathcal{B}/T^{-1}\mathcal{B})$ generates the full σ -algebra \mathcal{B} . If T is irregular then it is characterized by a countable number of invariants. Contrary to our expectations we have shown that irregular endomorphisms form a set of first category; in fact $I(\mathcal{B}/T^{-1}\mathcal{B}) = \infty$ a.e. is the general case.

Due to this failure we turned to an analogous problem. We study the set \mathcal{M}_g of so called g -measures (defined with respect to an aperiodic shift of finite type) with a natural topology stronger than the weak*-topology. In this topology the set of g -measures making the shift an exact endomorphism form a dense G_δ and the Markov measures are dense.

The irregular g -measures form an open dense set and can be characterized by a countable number of invariants. Also most g 's have a unique g -measure, but we are unable to decide if all g 's have a unique g -measure.(see[K]).

§1. The space $E(X)$.

Let (X, \mathcal{B}, m) be a non-atomic Lebesgue space i.e. a probability space isomorphic to the unit interval with Lebesgue measurable sets and Lebesgue measure. A measure-preserving transformation of (X, \mathcal{B}, m) is called an endomorphism of (X, \mathcal{B}, m) . The space (semi-group) of all endomorphisms of X will be denoted by $E(X)$. The subset (subgroup) of $E(X)$ consisting of all invertible endomorphisms (automorphisms) will be denoted by $A(X)$. The group $A(X)$ is usually endowed with the weak operator topology inherited from the weak topology on the group $U(L^2(X))$ of unitary operators on $L^2(X)$ by the injection $A(X) \ni T \rightarrow U_T$ where $U_T \in U(L^2(X))$ and $U_T f = f \circ T$. [H]. The injection $E(X) \ni T \rightarrow U_T$ associates an isometry U_T of $L^2(X)$ to each T . The weak and strong topologies coincide on the set of isometries of $L^2(X)$. We shall denote the adjoint of U_T by U_T^* ($U_T^* f(x) = E(f/T^{-1}\mathcal{B})(T^{-1}x)$) and we write $U_{T_n}^* \rightarrow U_T^*$ to denote convergence in the strong operator topology. We have $U_{T_n}^* \rightarrow U_T^*$ implies $U_{T_n} \rightarrow U_T$ (see proposition 1) but although $U_{T_n} \rightarrow U_T$ implies $U_{T_n}^* \rightarrow U_T^*$ is valid for automorphisms it is not true for endomorphisms. For this reason we consider the following two topologies on $E(X)$, (from now on we write Tf for $U_T f$).

(i) Weak Topology. A neighbourhood of $S \in E(X)$ is specified by a finite set f_1, \dots, f_k of members of $L^2(X)$ and some $\varepsilon > 0$:

$$U(S; f_1, \dots, f_k; \varepsilon) = \{T \in E(X) : \|Sf_i - Tf_i\| < \varepsilon, 1 \leq i \leq k\}.$$

A sequence $\{T_n\}$ converges to S in the weak topology if $\|T_n f - S f\| \rightarrow 0$ for all $f \in L^2(X)$.

(ii) Strong adjoint topology. A neighbourhood of $S \in E(X)$ is specified by a finite set f_1, \dots, f_k of members of $L^2(X)$ and some $\varepsilon > 0$:
 $U(S; f_1, \dots, f_k; \varepsilon) = \{T \in E(X) : \|Sf_i - Tf_i\| < \varepsilon \text{ and } \|S^*f_i - T^*f_i\| < \varepsilon \text{ } 1 \leq i \leq k\}$.
 A sequence $\{T_n\}$ converges to S in this topology if $\|T_n f - S f\| \rightarrow 0$
 and $\|T_n^* f - S^* f\| \rightarrow 0$ for all $f \in L^2(X)$.

Proposition 1.

Let $\{T_n\}$ be a sequence in $E(X)$ and let $S \in E(X)$. Then the statement " T_n converges to S in the strong adjoint topology" is equivalent to each of the following

- (a) $\|T_n^* f - S^* f\| \rightarrow 0$ for all $f \in L^2(X)$
 (b) $\|T_n f - S f\| \rightarrow 0$ and $\|E(f/T_n^{-1}\mathcal{B}) - E(f/S^{-1}\mathcal{B})\| \rightarrow 0$ for all $f \in L^2(X)$.

Proof

(a) We have to show $\|T_n^* f - S^* f\| \rightarrow 0$ for all $f \in L^2(X)$ implies $\|T_n f - S f\| \rightarrow 0$ for all $f \in L^2(X)$.

$$\begin{aligned} \text{If } f \in L^2(X) \quad \|T_n f - S f\|^2 &= 2\|f\|^2 - \langle T_n f, S f \rangle - \langle S f, T_n f \rangle \\ &= 2\|f\|^2 - \langle f, T_n^* S f \rangle - \langle T_n^* S f, f \rangle \\ &\rightarrow 2\|f\|^2 - \langle f, S^* S f \rangle - \langle S^* S f, f \rangle \\ &= 2\|f\|^2 - \langle S f, S f \rangle - \langle S f, S f \rangle = 0. \end{aligned}$$

(b) We have $E(f/T_n^{-1}\mathcal{B}) = T_n T_n^* f$ so

$$\begin{aligned} \|E(f/T_n^{-1}\mathcal{B}) - E(f/S^{-1}\mathcal{B})\| &= \|T_n T_n^* f - S S^* f\| \\ &\leq \|T_n(T_n^* f - S^* f)\| + \|(T_n - S)S^* f\| \\ &= \|T_n^* f - S^* f\| + \|(T_n - S)S^* f\| \end{aligned}$$

and

$$\begin{aligned} \|T_n^* f - S^* f\| &= \|T_n(T_n^* f - S^* f)\| \leq \|T_n T_n^* f - S S^* f\| + \|(S - T_n)S^* f\| \\ &= \|E(f/T_n^{-1}\mathcal{B}) - E(f/S^{-1}\mathcal{B})\| + \|(S - T_n)S^* f\|. \end{aligned}$$

We remark that it is sufficient to check (a) or (b) for $f = \chi_A$, $A \in \bigcup_k \hat{\beta}_k$, where $\{\beta_k\}$ is a sequence of finite partitions whose σ -algebras $\hat{\beta}_k$ increase to \mathcal{B} (i.e. $\hat{\beta}_k \uparrow \mathcal{B}$).

Proposition 2

Let $\{f_n\} \subset L^2(X)$ be a sequence of functions of norm 1 whose linear span is dense in $L^2(X)$. Then

$$D(S, T) = \sum_{n=1}^{\infty} 1/2^n (\|Sf_n - Tf_n\| + \|S^*f_n - T^*f_n\|) \quad \text{is a complete metric}$$

on $E(X)$ compatible with the strong adjoint topology.

Proof

The proof is the same as that for automorphisms [H]. //

We have already remarked that the two topologies (i) and (ii) coincide on $A(X)$. However

Proposition 3

$A(X)$ is dense in $E(X)$ with respect to the weak topology and is closed and nowhere dense with respect to the strong adjoint topology.

Proof

Let $\{\beta_n\}$ be an increasing sequence of finite partitions whose σ -algebras $\hat{\beta}_n$ increase to \mathcal{B} . To show $A(X)$ is dense in the weak topology it suffices to construct, for each $S \in E(X)$, $k \in \mathbb{Z}^+$, $\varepsilon > 0$, some $T \in A(X)$ with $\|T\chi_A - S\chi_A\| < \varepsilon$ for all $A \in \beta_k$. But this is immediate:- for each $A \in \beta_k$ define an invertible measure-preserving transformation of $S^{-1}A$ onto A , and the combined transformation T is an automorphism with $T\chi_A = S\chi_A$.

We shall now show that $A(X)$ is closed in the strong adjoint topology.

If $\{T_n\}$ is a sequence of automorphisms converging to $S \in E(X)$ then by Proposition 1 we have $E(f/T_n^{-1}\mathcal{B}) \rightarrow E(f/S^{-1}\mathcal{B})$ in $L^2(X)$ for each $f \in L^2(X)$. But $E(f/T_n^{-1}\mathcal{B}) = f$ for each n and therefore $E(f/S^{-1}\mathcal{B}) = f$ for all $f \in L^2(X)$. Hence $S \in A(X)$, and $A(X)$ is closed.

In §2 we shall show that the exact Markov endomorphisms are dense in $E(X)$ and therefore $A(X)$ has no interior. //

Proposition 4.

The strong-mixing endomorphisms form a set of first category with respect to the weak topology.

Proof

The proof for automorphisms is contained in [H] and goes over for our case on noting that $A(X)$ is dense in $E(X)$ in the weak topology (proposition 3). //

An endomorphism T is said to be exact if $\bigcap_0^\infty T^{-n}\mathcal{B} = \mathcal{N}$.

Exact endomorphisms are the "opposites" of automorphisms. As we have said, we shall prove that exact Markov endomorphisms are dense in $E(X)$ with respect to the strong adjoint topology. Assuming this for the moment we prove in contrast to Proposition 4:-

Theorem 1.

The set of exact endomorphisms is a dense G_δ in $E(X)$ with respect to the strong adjoint topology.

Proof

T is exact if and only if $E(f/\bigcap_0^\infty T^{-n}\mathcal{B}) = \int f \, d\mu$ for all

$f \in L^2(X)$. We have $E(f/T^{-n}\mathcal{B}) = T^n T^{*n} f$. Let $\{f_i\}^\omega$ be dense in $L^2(X)$.

For natural numbers r, i, n let $U_{r,i,n} = \{T \in E(X) : \|T^n T^{*n} f_i - \int f_i \, d\mu\| < 1/r\}$.

This is an open set in the strong adjoint topology and therefore

$$\bigcap_r \bigcap_i \bigcap_N \bigcup_{n \geq N} U_{r,i,n} \text{ is a } G_\delta.$$

We claim that this G_δ set is precisely the set of exact endomorphisms.

By the Martingale theorem we know $\|E(f/T^{-n}\mathcal{B}) - E(f/\bigcap_0^\infty T^{-j}\mathcal{B})\| \rightarrow 0$

for all $f \in L^2(X)$. Therefore every exact endomorphism belongs to the

G_δ set. Conversely, if T belongs to the G_δ set then for each i and

each N there exists $n_N \geq N$ with

$$\|E(f_i/T^{-n_N}\mathcal{B}) - \int f_i \, d\mu\| < 1/N. \text{ Therefore } \|E(f_i/T^{-n_N}\mathcal{B}) - \int f_i \, d\mu\| \rightarrow 0$$

as $N \rightarrow \infty$ and we must have $\int f_i \, d\mu = E(f_i/\bigcap_0^\infty T^{-n}\mathcal{B})$. Since $\{f_i\}$ is a

dense subset of $L^2(X)$ we have that T is exact. //

§2. Exact Markov Endomorphisms are Dense.

If A, B are matrices of the same size we will write $|A-B|$ for the maximum absolute value of the entries of $A-B$.

Let P be a stochastic $k \times k$ matrix with ^{\dagger} strictly positive left fixed vector p , the sum of whose elements equals 1. Such a matrix cannot have trivial rows or columns and will be called non-trivial. It is well known that up to a permutation equivalence P may be written

$$P = \begin{pmatrix} A & & \\ & B & \\ & C & \\ & & \ddots \end{pmatrix}$$

where A, B, C, \dots are irreducible and all other entries are zeros. We write, correspondingly, $p = (a, b, c, \dots)$. For convenience of presentation we shall assume that P is composed of three such irreducible matrices.

Lemma 1. For any $\varepsilon > 0$ there exists an aperiodic irreducible stochastic matrix Q with strictly positive left fixed vector q (whose elements sum to 1) such that $|P-Q| < \varepsilon$, $|p-q| < \varepsilon$.

Proof. The matrix A , being irreducible, has a unique left fixed vector a whose entries sum to $\|a\|$. The compactness of stochastic matrices and vectors with a given sum implies that there exist strictly positive matrices A' with left fixed vector a' as close as we like to A , a respectively such that $\|a'\| = \|a\|$. The same is true, of course, for B and C . To prove the lemma, then, we may assume without loss of generality that A, B, C are strictly positive. Let $P = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$ be

written as

$$\left(\begin{array}{ccc|ccc|ccc} a_{11} & \dots & a_{1r} & & & & & & \\ & & & & & & & & \\ \hline a_{r1} & \dots & a_{rr} & & & & & & \\ & & & b_{11} & \dots & b_{1s} & & & \\ & & & & & & & & \\ \hline & & & b_{s1} & \dots & b_{ss} & & & \\ & & & & & & c_{11} & \dots & c_{1t} \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & c_{t1} & \dots & c_{tt} \end{array} \right)$$

and let

$$P_{\varepsilon} = \left(\begin{array}{ccc|ccc|ccc} a_{11} - \rho & \dots & a_{1r} & \rho & & & & & \\ & & & & & & & & \\ \hline a_{r1} & \dots & a_{rr} & & & & & & \\ & & & b_{11} - \sigma & \dots & b_{1s} & \sigma & & \\ & & & & & & & & \\ \hline & & & b_{s1} & \dots & b_{ss} & & & \\ & & \gamma & & & & c_{11} - \tau & \dots & c_{1t} \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & c_{t1} & \dots & c_{tt} \end{array} \right) .$$

The subtractions above occur only in 3 places and are compensated for (to keep P_{ε} stochastic) in 3 places.

Suppose $a = (a_1, \dots, a_r)$, $b = (b_1, \dots, b_s)$ and $c = (c_1, \dots, c_t)$. Then

$$(abc)P_{\varepsilon} = (a_1(1-\rho) + \tau c_1, a_2, \dots, a_r; b_1(1-\sigma) + \rho a_1, b_2, \dots, b_s; c_1(1-\tau) + b_1\sigma, c_2, \dots, c_t)$$

so $pP_{\varepsilon} = p$ if

$$a_1(1-\rho) + \tau c_1 = a_1, \quad b_1(1-\sigma) + \rho a_1 = b_1, \quad c_1(1-\tau) + b_1\sigma = c_1.$$

Hence $pP_{\varepsilon} = p$ if $\tau c_1 = a_1\rho$, $a_1\rho = b_1\sigma$, $b_1\sigma = c_1\tau$.

We therefore choose $\tau < \max(\varepsilon, a_1\varepsilon/c_1, b_1\varepsilon/c_1)$ and define

$$\rho = \tau c_1 / a_1, \quad \sigma = \tau c_1 / b_1.$$

Then $|P - P_{\varepsilon}| < \varepsilon$ and $pP_{\varepsilon} = p$. //

Lemma 2. Let $\{\beta_n\}$ be an increasing sequence of finite partitions such that $\hat{\beta}_n \uparrow \mathcal{B}$. The sets

$$\mathcal{S}(k, n, \delta, \varepsilon) = \{T \in E(X) : \|E(\chi_A / S^{-1} \hat{\beta}_n) - E(\chi_A / T^{-1} \mathcal{B})\| < \varepsilon$$

for all $A \in \beta_k$ and $\|S\chi_B - T\chi_B\| < \delta$ for all $B \in \beta_n\}$ for $k < n \in \mathbb{Z}^+$,

$\delta > 0, \varepsilon > 0$, form a fundamental system of neighbourhoods of S in the strong adjoint topology.

Proof. From proposition 1 we know $\{T : \|S^* \chi_A - T^* \chi_A\| < \varepsilon \text{ for all } A \in \beta_k\}$ form a fundamental system of neighbourhoods of S in the strong adjoint topology. Let $k \in \mathbb{Z}^+$ and $\varepsilon > 0$ be given.

$$\begin{aligned} \|S^* \chi_A - T^* \chi_A\| &= \|TS^* \chi_A - TT^* \chi_A\| \\ &\leq \|TS^* \chi_A - SS^* \chi_A\| + \|SS^* \chi_A - TT^* \chi_A\| \\ &= \|TS^* \chi_A - SS^* \chi_A\| + \|E(\chi_A / S^{-1} \mathcal{B}) - E(\chi_A / T^{-1} \mathcal{B})\| \\ &\leq \|TS^* \chi_A - SS^* \chi_A\| + \|E(\chi_A / S^{-1} \hat{\beta}_n) - E(\chi_A / T^{-1} \mathcal{B})\| + \varepsilon/3 \end{aligned}$$

if n is large enough.

Choose $\delta > 0$ and choose $n > k$ so that the above inequality holds and also so that $\|T\chi_B - S\chi_B\| < \delta$ for all $B \in \beta_n$ implies

$$\|TS^* \chi_A - SS^* \chi_A\| < \varepsilon/3 \text{ for all } A \in \beta_k.$$

Then $\mathcal{S}(k, n, \delta, \varepsilon/3) \subset \{T : \|S^* \chi_A - T^* \chi_A\| < \varepsilon\}$. //

We shall call $T \in E(X)$ a Markov endomorphism if it is isomorphic to the one-sided shift on a stationary Markov chain with a finite number of states.

Theorem 2. Exact Markov endomorphisms are dense in $E(X)$ with respect to the strong adjoint topology.

Proof. Let $S \in \mathcal{E}(X)$ and let $\{\beta_n\}$ be an increasing sequence of finite partitions so that $\hat{\beta}_n \uparrow \mathcal{B}$. Fix $\varepsilon > 0$, $\delta > 0$ and $k, n \in \mathbb{Z}^+$ with $k < n$. We shall construct an exact Markov endomorphism T in the neighbourhood $\mathcal{F}(k, n, \delta, \varepsilon)$ of S (see lemma 2).

Let $\beta_n = \{B_1, \dots, B_N\}$ where $m(B_i) > 0$ all i , and define the non-trivial stochastic matrix P by $P(i, j) = m(B_i \cap S^{-1}B_j) / m(B_i)$.

P has a left fixed probability vector p ($pP = p$) given by $p(i) = m(B_i)$. Let $\eta > 0$ be chosen later and, using lemma 1, let \tilde{P} be an irreducible aperiodic stochastic matrix with left fixed probability vector \tilde{p} such that $|P - \tilde{P}| < \eta$ and $|p - \tilde{p}| < \eta$. Let \tilde{m} denote the shift invariant Markov probability measure defined on $\tilde{X} = \prod_{n=0}^{\infty} \{1, \dots, N\}$ by \tilde{P} , \tilde{p} , and let \tilde{S} denote the shift on \tilde{X} .

If $[i, j] = \{\tilde{x} : \tilde{x}_0 = i, \tilde{x}_1 = j\}$ then $\tilde{m}([i, j]) = \tilde{p}(i)\tilde{P}(i, j)$ so that $|m(B_i \cap S^{-1}B_j) - \tilde{m}([i, j])| < 2\eta$. Let $\phi: X \rightarrow \tilde{X}$ be an isomorphism which maps $B_i \cap S^{-1}B_j$ into $[i, j]$ if $m(B_i \cap S^{-1}B_j) \leq \tilde{m}([i, j])$ and whose inverse maps $[i, j]$ into $B_i \cap S^{-1}B_j$ otherwise.

Now put $T = \phi^{-1}\tilde{S}\phi$. We have

$$\|\chi_{B_j} - \chi_{\phi^{-1}[j]}\| = \left\| \sum_k (\chi_{B_j \cap S^{-1}B_k} - \chi_{\phi^{-1}[j, k]}) \right\| \leq N(2\eta)^{\frac{1}{2}}, \text{ and}$$

$$\begin{aligned} \|\chi_{B_j} \circ S - \chi_{B_j} \circ T\| &= \left\| \sum_i (\chi_{B_i \cap S^{-1}B_j} - \chi_{B_i \cap T^{-1}B_j}) \right\| \\ &\leq \left\| \sum_i (\chi_{B_i \cap S^{-1}B_j} - \chi_{\phi^{-1}[i, j]}) \right\| + \|\chi_{\phi^{-1}[j]} \circ T - \chi_{B_j} \circ T\| \\ &\leq 2N(2\eta)^{\frac{1}{2}}. \end{aligned}$$

Therefore

$$\begin{aligned}
& \|E(\chi_{B_i}/S^{-1}\beta_n) - E(\chi_{B_i}/T^{-1}\mathcal{B})\| \\
& \leq \| \sum_j \chi_{B_j} \circ S \cdot m(B_i \cap S^{-1}B_j)/m(B_j) - E(\chi_{\phi^{-1}[i]}/T^{-1}\mathcal{B}) \| \\
& \quad + \|E((\chi_{\phi^{-1}[i]} - \chi_{B_i})/T^{-1}\mathcal{B})\| \\
& \leq \| \sum_j \chi_{B_j} \circ S \cdot m(B_i \cap S^{-1}B_j)/m(B_j) - \sum_j \chi_{\phi^{-1}[j]} \circ T \tilde{p}(i)\tilde{P}(i,j)/\tilde{p}(j) \| \\
& \quad + \| \chi_{\phi^{-1}[i]} - \chi_{B_i} \| \\
& \leq \sum_j \| \chi_{B_j} \circ S - \chi_{\phi^{-1}[j]} \circ T \| + \sum_j |p(i)P(i,j)/p(j) - \tilde{p}(i)\tilde{P}(i,j)/\tilde{p}(j)| \\
& \quad + \| \chi_{\phi^{-1}[i]} - \chi_{B_i} \| \\
& \leq \sum_j \| \chi_{B_j} \circ S - \chi_{B_j} \circ T \| + \sum_j \| \chi_{B_j} \circ T - \chi_{\phi^{-1}[j]} \circ T \| + \\
& \quad \sum_j \left[1/p(j) |p(i)P(i,j) - \tilde{p}(i)\tilde{P}(i,j)| + \tilde{p}(i)\tilde{P}(i,j) |1/p(j) - 1/\tilde{p}(j)| \right] \\
& \quad + N(2\eta)^{\frac{1}{2}} \\
& \leq 3N^2(2\eta)^{\frac{1}{2}} + N(2\eta)^{\frac{1}{2}} + 2\eta \sum_j 1/p(j) + \eta \sum_j 1/(p(j)[p(j) - \eta]).
\end{aligned}$$

Hence if $A \in \beta_k$ then A is a union of some members of β_n and

$$\begin{aligned}
\|E(\chi_A/S^{-1}\hat{\beta}_n) - E(\chi_A/T^{-1}\mathcal{A})\| & \leq N \left(3N^2(2\eta)^{\frac{1}{2}} + N(2\eta)^{\frac{1}{2}} + \right. \\
& \quad \left. + 2\eta \sum_j 1/p(j) + \eta \sum_j 1/p(j)[p(j) - \eta] \right).
\end{aligned}$$

Now choose $\eta > 0$ so that this latter quantity is less than ε

(and $\eta < \min_j m(B_j)$) and so that $2N(2\eta)^{\frac{1}{2}} < \delta$. Then $T \in \mathcal{S}(k, n, \delta, \varepsilon)$.

T is exact because one-sided Markov shifts on aperiodic irreducible Markov chains are exact $[B, F]$.

//

§3 Irregular Endomorphisms.

We say that a $k \times k$ irreducible stochastic matrix P , with invariant initial probability p , is irregular if the function on $\prod_{n=0}^{\infty} \{1, \dots, k\}$ defined by $x \rightarrow p(x_0)P(x_0 x_1)/p(x_1)$ generates (up to sets of measure zero) the full σ -algebra. This is the case, of course, if the matrix P' , given by $P'(i, j) = p(i)P(i, j)/p(j)$, satisfies $P'(i, j) \neq P'(k, l)$ when $(i, j) \neq (k, l)$.

Clearly irregularity is a generic property for $k \times k$ matrices. This may suggest that irregularity is generic among all endomorphisms if we define $T \in E(X)$ to be irregular if $E_T = E(\mathcal{B}/T^{-1}\mathcal{B})$ generates the full σ -algebra. (If α is a finite partition then

$$E(\alpha/T^{-1}\mathcal{B}) = \sum_{A \in \alpha} \chi_{A^m(A/T^{-1}\mathcal{B})} \text{ and } E(\mathcal{B}/T^{-1}\mathcal{B}) = \lim_{n \rightarrow \infty} E(\beta_n/T^{-1}\mathcal{B})$$

for any increasing sequence $\{\beta_n\}$ of finite partitions with $\hat{\beta}_n \uparrow \mathcal{B}$.

$I(\mathcal{B}/T^{-1}\mathcal{B}) = -\log E(\mathcal{B}/T^{-1}\mathcal{B})$ is the information function of T . [P]).

An irregular endomorphism T is characterized up to isomorphism by the invariants

$$\chi_n(s_1, \dots, s_n) = \int \exp 2\pi i(s_1 E_T(x) + \dots + s_n E_T(T^{n-1}x)) \, d\mu, \quad n = 1, 2, \dots$$

because T is isomorphic to the shift on $\prod_{n=0}^{\infty} \mathbb{R}$ endowed with the

probability m_T given by $m_T(F) = m(\{x: (E_T(x), E_T(Tx), \dots) \in F\})$.

This measure is characterized by the n -fold characteristic functions $\chi_n, n \geq 1$.

A slight modification of our proof that exact Markov endomorphisms are dense in $E(X)$ shows that irregular exact Markov endomorphisms are also dense. However irregularity is not generic. In fact

Theorem 3. The set of endomorphisms T with $E_T = 0$ a.e.

(or $I_T = I(\mathcal{B}/T^{-1}\mathcal{B}) = \infty$ a.e.) is a dense G_δ , so that irregular endomorphisms form a set of first category.

Proof. The set of endomorphisms T with $E_T = 0$ a.e. is

$$\bigcap_n \bigcup_{\alpha \text{ finite}} \left\{ T: \int E(\alpha/T^{-1}\beta) \, dm < 1/n \right\}.$$

Since $E(\chi_A/T^{-1}\beta) = TT^*\chi_A$, the above set is a G_δ . We have to prove that the above is dense.

Let S be an exact Markov endomorphism with generator α so that $\alpha^n = \alpha \vee \dots \vee S^{-n}\alpha$ has the property that $\hat{\alpha}^n \uparrow \beta$. We shall find endomorphisms T arbitrarily close to S with the property that $E(\beta/T^{-1}\beta) = 0$. This will complete the proof by virtue of Theorem 2. It suffices, for arbitrarily large n , to produce such an endomorphism T with the property that $T\chi_A = S\chi_A$ and $E(\chi_A/S^{-1}\beta) = E(\chi_A/T^{-1}\beta)$ for all $A \in \alpha^n$. Since S is Markov with respect to the generator α^n , there is no loss in generality, if we do this for \wedge arbitrary Markov generator, for which we shall retain the symbol α .

Let S_1 be an endomorphism of a Lebesgue space (X_1, β_1, m_1) with the property that $I(\beta_1/S_1^{-1}\beta_1) = \infty$ a.e. For example S_1 could be the Bernoulli endomorphism with unit interval as state space.

Define $(\tilde{X}, \tilde{\beta}, m) = (X, \beta, m) \times (X_1, \beta_1, m_1)$ and $\tilde{T} = S \times S_1$.

Corresponding to the partition $\{A \cap S^{-1}B : A, B \in \alpha\}$ of X we have a partition $\{(A \cap S^{-1}B) \times X_1 : A, B \in \alpha\}$ of \tilde{X} . Let ϕ be an isomorphism of X onto \tilde{X} which maps $A \cap S^{-1}B$ onto $(A \cap S^{-1}B) \times X_1$ when $A, B \in \alpha$.

Now define $Tx = \phi^{-1}\tilde{T}\phi(x)$ so that $T^{-1}B = S^{-1}B$ and hence $\chi_B \circ S = \chi_B \circ T$

for $B \in \alpha$. Finally $E(\chi_B/S^{-1}\beta) = E(\chi_B/S^{-1}\hat{\alpha})$ when $B \in \alpha$ and

$$E(\chi_B/T^{-1}\beta) = E(\chi_{\phi(B)}/\phi T^{-1}\beta) \circ \phi = E(\chi_{B \times X_1}/S^{-1}(\beta) \times (\beta_1)) \circ \phi$$

$$= E(\chi_B/S^{-1}\beta) \circ \pi_1 \circ \phi \quad (\text{where } \pi_1(x, x_1) = x)$$

$$= E(\chi_B/S^{-1}\hat{\alpha}) \circ \pi_1 \circ \phi = E(\chi_B \circ \pi_1 \circ \phi / \phi^{-1} \circ \pi_1^{-1} \circ S^{-1}\hat{\alpha})$$

$$\begin{aligned}
&= E(\chi_{\phi^{-1}(B \times X_1)} / \phi^{-1}(S^{-1}\hat{\alpha} \times X_1)) = E(\chi_B / S^{-1}\hat{\alpha}) \\
&= E(\chi_B / S^{-1}B).
\end{aligned}$$

All that remains is to show that $I(B/T^{-1}B) = \infty$ a.e.

But $I(\tilde{B}/\tilde{T}^{-1}\tilde{B}) \circ \phi = I(B/T^{-1}B)$ and

$$I(\tilde{B}/\tilde{T}^{-1}\tilde{B})(x, x_1) = I(B/S^{-1}B)(x) + I(B_1/S^{-1}B_1)(x_1) = \infty \quad \text{a.e.} //$$

The Bernoulli endomorphism which has the unit interval (with Lebesgue measure) as its state space will be called the uniform Bernoulli endomorphism.

Corollary The set of exact endomorphisms with infinite entropy contains a dense G_δ . The set of exact endomorphisms having the uniform Bernoulli endomorphism as a factor contains a dense G_δ .

Proof Theorems 1 and 3 show that the set of exact endomorphisms T of X with $I(B/T^{-1}B) = \infty$ a.e. form a dense G_δ . Such endomorphisms have the uniform Bernoulli endomorphism as a factor because Rohlin's measure theory [R] gives a non-atomic σ -algebra \mathcal{C} as an independent complement to $T^{-1}B$ (i.e. is independent of $T^{-1}B$ and $\mathcal{C} \vee T^{-1}B = \mathcal{B}$). //

If T is an exact endomorphism and $\mathcal{B} = \mathcal{C} \vee T^{-1}B$ where \mathcal{C} is non-atomic and \mathcal{C} and $T^{-1}B$ are independent, it is natural to ask if T is isomorphic to the uniform Bernoulli endomorphism. This kind of problem appeared in [Wi] where the impression is given that the answer is affirmative. However the answer is negative as the following reasoning shows. Take an exact endomorphism S that does not have a Bernoulli natural extension. (S exists because Ornstein [O] has shown the existence of Kolmogorov automorphisms which are not Bernoulli shifts.) Let V denote the uniform Bernoulli endomorphism and put $T = S \times V$. Then $I(B/T^{-1}B) = \infty$ so that a non-atomic \mathcal{C} exists with $\mathcal{C} \vee T^{-1}B = \mathcal{B}$ and \mathcal{C} independent of $T^{-1}B$. However T is not isomorphic to V because any factor of the uniform Bernoulli endomorphism has a Bernoulli natural extension [O].

Rosenblatt [Ros] has discussed related problems when \mathcal{C} is atomic. The answer is negative in this case too. (see also [P.W]).

§ 4. g -measures.

In this section we consider some problems analogous to those of the earlier sections. Rather than fixing a measure and considering all transformations preserving it, we shall consider a fixed continuous transformation $T:X \rightarrow X$ of a compact space and study all the T -invariant measures on X whose information function is continuous in a strong sense.

We shall study the case where $T:X \rightarrow X$ is a one-sided subshift of finite type. This means there is a finite set C , with $|C|$ elements, and a $|C| \times |C|$ matrix A whose entries are zeros and ones so that X is the subset of $C^{\mathbb{Z}^+}$ defined by $x = \{x_n\}_0^\infty \in X$ if and only if $A(x_n, x_{n+1}) = 1$ for all $n \geq 0$. If C is given the discrete topology and $C^{\mathbb{Z}^+}$ the product topology then X is a closed subset of the compact metrisable space $C^{\mathbb{Z}^+}$. $T:X \rightarrow X$ is defined by $(Tx)_n = x_{n+1}$, $n \geq 0$. T is a local homeomorphism. We shall always assume T is topologically mixing, which is equivalent to assuming there exists $n > 0$ with A^n having all entries strictly positive.

$$\text{Let } \mathcal{G} = \left\{ g \in C(X) : g > 0 \text{ and } \sum_{y \in T^{-1}x} g(y) = 1 \text{ for all } x \in X \right\}.$$

If we equip \mathcal{G} with the metric ρ defined by $\rho(g_1, g_2) = \|\log g_1 - \log g_2\|_\infty$ ($\|\cdot\|_\infty$ denotes the supremum norm on $C(X)$), then \mathcal{G} is a complete metric space. For $g \in \mathcal{G}$ we can define $L_g : C(X) \rightarrow C(X)$ by $L_g f(x) = \sum_{y \in T^{-1}x} g(y) f(y)$. Since L_g is a positive operator and $L_g(1) = 1$, its dual L_g^* maps the compact convex set $\mathcal{M}(X)$ of all Borel probabilities on X into itself. Therefore L_g^* always has at least one fixed point in $\mathcal{M}(X)$. Any member of $\mathcal{M}(X)$ which is a fixed point of L_g^* is called a g -measure, [K]. These measures are important in the study of equilibrium states [W]. Let $\mathcal{M}(T)$ denote those members of $\mathcal{M}(X)$ which are T -invariant.

Lemma 3. [L], [W] .

- (i) μ is a g -measure if and only if $\mu \in \mathcal{M}(T)$ and $I_\mu(\mathcal{B}/T^{-1}\mathcal{B}) = -\log g$
- (ii) μ is a g -measure if and only if $\mu \in \mathcal{M}(T)$ and $h_\mu(T) = -\int \log g \, d\mu$
- (iii) A g -measure has support X .

Let \mathcal{M}_g denote the collection of all g -measures as g runs through \mathcal{G} . We have a natural map $\pi: \mathcal{M}_g \rightarrow \mathcal{G}$ given by $\pi(\mu) = g$ if μ is a g -measure. It is unknown if this map π is injective or not. However it is known that $\pi^{-1}(g)$ is a singleton for g in a dense subset of \mathcal{G} , ([K], [W]). We shall show in theorem 5 that

$\{g \in \mathcal{G} : \pi^{-1}(g) \text{ is a singleton}\}$ contains a dense $\bar{\mathcal{G}}_g$. If \mathcal{G}_k denotes the set of those g depending only on the first k coordinates (i.e. k is the least natural number with the property $g(x) = g(y)$ if $x_i = y_i$ $0 \leq i \leq k-1$) then $\pi^{-1}(g)$ is a singleton for $g \in \mathcal{G}_k$. $\bigcup_k \mathcal{G}_k$ is dense in \mathcal{G} . If μ is a g -measure for $g \in \mathcal{G}_k$ then μ is a $k-1$ step Markov measure. We shall call the members of this family $(\pi^{-1}(\bigcup_k \mathcal{G}_k))$ strong Markov measures. They are supported on X .

Let d denote a metric on $\mathcal{M}(T)$ giving the weak*-topology. $\mathcal{M}(T)$ is compact with this topology. Define a metric D on \mathcal{M}_g by

$$D(\mu, \nu) = d(\mu, \nu) + \rho(\pi(\mu), \pi(\nu)).$$

Lemma 4. \mathcal{M}_g is complete with respect to the metric D .

Proof. Suppose $\{\mu_n\}$ is a Cauchy sequence for D . Then $\{\mu_n\}$ is Cauchy for d and hence $\mu_n \xrightarrow{d} \mu \in \mathcal{M}(T)$. Also $\|\log g_n - \log g\|_\infty \rightarrow 0$ for some $g \in \mathcal{G}$. It remains to show μ is a g -measure and this follows

because

$$\int L_g h \, d\mu = \lim_{n \rightarrow \infty} \int L_{g_n} h \, d\mu_n = \lim_{n \rightarrow \infty} \int h \, d\mu_n = \int h \, d\mu \text{ for all } h \in C(X). //$$

The map $\pi: \mathcal{M}_g \rightarrow \mathcal{G}$ is clearly continuous. The topology on \mathcal{M}_g given by D is strictly stronger than that given by d .

We shall illustrate this when $X = \{0,1\}^{\mathbb{Z}^+}$.

Define g_n by $g_n(x) = \begin{cases} \frac{3}{4} & \text{if } (x_0, \dots, x_{n-1}) = (0, 0, \dots, 0) \\ \frac{1}{4} & \text{if } (x_0, \dots, x_{n-1}) = (1, 0, 0, \dots, 0) \\ \frac{1}{2} & \text{otherwise.} \end{cases}$

If μ_n is the unique g_n -measure then μ_n is a $(n-1)$ -step Markov measure and one can show that $\mu_n \xrightarrow{d} \mu$ where μ is the product measure with weights $(\frac{1}{2}, \frac{1}{2})$. μ is the g -measure for $g = \frac{1}{2}$ but $\|\log g_n - \log \frac{1}{2}\|_{\infty} \not\rightarrow 0$.

We shall denote the cylinder set $\{x: x_i = a_i, 0 \leq i \leq n-1\}$ by $[a_0, a_1, \dots, a_{n-1}]$.

Proposition 5. Let μ be a g -measure and define $g_n \in \mathcal{G}$ by

$g_n(x) = \mu([x_0, x_1, \dots, x_{n-1}]) / \mu([x_1, \dots, x_{n-1}])$. Then

$\|\log g_n - \log g\|_{\infty} \rightarrow 0$.

Proof. Since g is uniformly continuous we have

$c_n = \sup \{g(w)/g(z): w_i = z_i, 0 \leq i \leq n-1\} \rightarrow 1$ as $n \rightarrow \infty$.

Since $d\mu/d\mu = 1/g$ we have $\mu([x_1, \dots, x_{n-1}]) = \int_{[x_0, \dots, x_{n-1}]} 1/g(z) d\mu(z)$

and therefore $g(x)/c_n \leq \mu([x_0, \dots, x_{n-1}]) / \mu([x_1, \dots, x_{n-1}]) \leq c_n g(x)$.

This gives $\|g_n - g\|_{\infty} \rightarrow 0$ and $\|\log g_n - \log g\|_{\infty} \rightarrow 0$. //

Proposition 6. The strong Markov measures are dense in \mathcal{M}_g (using the metric D).

Proof. Let μ be a g -measure and define g_n as above. Let μ_n

be the unique g_n -measure. It remains to show $\mu_n \xrightarrow{d} \mu$. It suffices

to show that for any given cylinder $[a_0, \dots, a_k]$, $\mu_n([a_0, \dots, a_k]) \rightarrow \mu([a_0, \dots, a_k])$. This follows because if $n > k$ we have

$\mu_n([a_0, \dots, a_k]) = \mu([a_0, \dots, a_k])$. //

Let us call g irregular if $\{g(T^n)\}_{n=0}^{\infty}$ separates points of X and call $\mu \in \mathcal{M}_g$ irregular if $\pi(\mu)$ is irregular. It is clear that if $g \in \mathcal{G}_k$ and g takes distinct values on the cylinders of length k then g is irregular.

Proposition 7. The irregular strong Markov measures are dense in \mathcal{M}_g (using the metric D).

Proof. If μ is a g -measure for $g \in \mathcal{G}_k$ then approximate g by some $g' \in \mathcal{G}_k$ with distinct values on k -cylinders. The unique g' -measure μ' will be close to μ in the D metric. //

Theorem 4. The subset of \mathcal{M}_g consisting of irregular measures contains an open dense set (using the metric D).

Proof. Let $g_0 \in \mathcal{G}_k$ take distinct values on k -cylinders. We shall show there is a neighbourhood U of g_0 in \mathcal{G} which consists of irregular g 's. Then $\pi^{-1}(U)$ is open in \mathcal{M}_g and the proof will be complete.

Let $\varepsilon = \min \{ |\log g_0([x_0, \dots, x_{k-1}]) - \log g_0([z_0, \dots, z_{k-1}])| : [x_0, \dots, x_{k-1}] \neq [z_0, \dots, z_{k-1}] \} > 0$.

Let $U = \{g \in \mathcal{G} : \rho(g, g_0) < \varepsilon/4\}$. Let $g \in U$.

If $x \neq z$ there is some j with $g_0(T^j x) \neq g_0(T^j z)$ and so

$$|\log g(T^j x) - \log g(T^j z)| \geq |\log g_0(T^j x) - \log g_0(T^j z)| - \varepsilon/2 \geq \varepsilon/2.$$

Therefore each $g \in U$ is irregular. //

If μ is an irregular g -measure then the map

$x \mapsto \phi_g(x, g(x), g(Tx), \dots)$ is a homeomorphism of X onto a closed

subset of $\prod_{n=0}^{\infty} [0, 1]$ which conjugates T with the shift on $\prod_{n=0}^{\infty} [0, 1]$.

The map ϕ_g takes μ to a measure on $\prod_{n=0}^{\infty} [0, 1]$ which is characterized by its n -fold characteristic functions (see § 3). Therefore an

open dense set of g -measures are characterized by a countable number of invariants. If μ' is an irregular g' -measure and μ, μ' have the same invariants then $\phi_{g'}^{-1} \circ \phi_g: X \rightarrow X$ is a homeomorphism mapping μ to μ' and commuting with T . Therefore any two irregular members of \mathcal{M}_g with the same invariants are related by a homeomorphism of X commuting with T .

We next show that 'most' $g \in \mathcal{G}$ have a unique g -measure and that 'most' $\mu \in \mathcal{M}_g$ are exact.

Theorem 5. $\{g \in \mathcal{G} : \text{there is a unique } g\text{-measure}\}$ contains a dense G_δ in \mathcal{G} .

Proof. Let $\{f_n\}_1^\infty$ be dense in $C(X)$. For natural numbers n, m, N and $c \in \mathbb{R}$ let $U_{n,m,c,N} = \{g \in \mathcal{G} : \|L_{g,n}^N f_n - c\|_\infty < 1/m\}$. This is an open subset of \mathcal{G} and therefore $\tilde{\mathcal{G}} = \bigcap_n \bigcap_m \bigcup_{c \in \mathbb{R}} \bigcup_{N \in \mathbb{N}} U_{n,m,c,N}$ is a

G_δ . We claim that

$$\tilde{\mathcal{G}} = \{g \in \mathcal{G} : \text{for all } f \in C(X) \text{ there exists } c(f) \in \mathbb{R} \text{ with } \|L_g^i f - c(f)\|_\infty \rightarrow 0\}.$$

If g belongs to this set then $g \in \tilde{\mathcal{G}}$. Conversely if $g \in \tilde{\mathcal{G}}$ then for all n, m there exists $c_m(n)$ and there exists N such that

$$\|L_{g,n}^N f_n - c_m(n)\|_\infty < 1/m. \text{ Since } \|L_g\|_\infty \leq 1 \text{ we have}$$

$\|L_{g,n}^i f_n - c_m(n)\|_\infty < 1/m$ for all $i \geq N$. If μ is any g -measure then $|\int f_n d\mu - c_m(n)| < 1/m$ so $\|L_{g,n}^i f_n - \int f_n d\mu\|_\infty < 2/m$ for all $i \geq N$, and for each n $\|L_{g,n}^i f_n - \int f_n d\mu\|_\infty \rightarrow 0$ as $i \rightarrow \infty$.

Therefore $\|L_g^i f - \int f d\mu\|_\infty \rightarrow 0$, for all $f \in C(X)$. Each $g \in \tilde{\mathcal{G}}$

has a unique g -measure because if $\|L_g^i f - c(f)\|_\infty \rightarrow 0$ then

$$\int f d\mu = c(f) \text{ for each } g\text{-measure } \mu. \text{ Since } \bigcup_{k=1}^\infty \mathcal{G}_k \subset \tilde{\mathcal{G}} \text{ ([W])}$$

we know that $\tilde{\mathcal{G}}$ is dense in \mathcal{G} . //

If $\mu \in \mathcal{M}(T)$ is such that T is an exact endomorphism relative to μ then we will say μ is exact.

Theorem 6. $\{\mu \in \mathcal{M}_g: \mu \text{ is exact}\}$ contains a dense G_δ in \mathcal{M}_g (with respect to D).

Proof. Let $\{f_n\}_{n=1}^\infty$ be dense in $C(X)$. For natural numbers n, m, N

let $U_{n,m,N} = \{\mu \in \mathcal{M}_g: \|L_{\pi(\mu)}^N f_n - \int f_n d\mu\|_\infty < 1/m\}$. $U_{n,m,N}$

is an open subset of \mathcal{M}_g and therefore $V = \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} U_{n,m,N}$ is a G_δ .

Using the fact that $\|L_{\pi(\mu)}\|_\infty \leq 1$ we have

$$V = \{\mu \in \mathcal{M}_g: \|L_{\pi(\mu)}^i f - \int f d\mu\|_\infty \rightarrow 0 \text{ for all } f \in C(X)\}.$$

We claim that each $\mu \in V$ is exact. If $\mu \in V$ then

$$\|L_{\pi(\mu)}^n f - \int f d\mu\|_\infty \rightarrow 0 \text{ for all } f \in L^1(\mu) \text{ and therefore}$$

$$\begin{aligned} \int |E_\mu(f/\bigcap_0^\infty T^{-n}\mathcal{B}) - \int f d\mu| d\mu &= \lim_{n \rightarrow \infty} \int |E_\mu(f/T^{-n}\mathcal{B}) - \int f d\mu| d\mu = \\ &= \lim_{n \rightarrow \infty} \int |L_{\pi(\mu)}^n f - \int f d\mu| d\mu = \lim_{n \rightarrow \infty} \int \|L_{\pi(\mu)}^n f - \int f d\mu\|_\infty d\mu = 0, \end{aligned}$$

for all $f \in L^1(\mu)$.

This shows that $\bigcap_0^\infty T^{-n}\mathcal{B}$ is trivial relative to μ and hence that μ is exact. We know the strong Markov measures are in V and these are dense by proposition 6. //

SECTION 2.

Mixing Properties of Linear Mod 1 Maps of the Unit Interval.

§0. Introduction.

Let $T_{\beta\alpha}(x) = \beta x + \alpha \bmod 1$ for $x \in [0, 1)$ and $T_{\beta\alpha}(1) = \lim_{x \uparrow 1} T_{\beta\alpha}(x)$, where $\beta > 1$ and $0 \leq \alpha < 1$. It has been shown that $T_{\beta\alpha}$ has a unique invariant probability measure $\mu_{\beta\alpha}$ absolutely continuous with respect to Lebesgue measure 1, [P 1], [C]. Such transformations are the 'simplest' of a larger class of piecewise C^2 and monotonically increasing functions of $[0, 1]$ which also have smooth invariant measures [L.Y]. The mixing properties of functions f in this class have been studied for $\beta \geq 2$ (where $\beta = \inf f'(x)$) and in some cases for $\beta \geq 2^{\frac{1}{2}}$, [W 1], [B].

In this paper we study the spectral and mixing properties of $T_{\beta\alpha}$ for $1 < \beta < 2$ and $0 < \alpha < 1$, determining the support of the measure $\mu_{\beta\alpha}$ (see also [H 2]), the subclass of weak Bernoulli (WB) transformations and completely determining the spectral properties of those transformations which are not WB.

Henceforth we shall write T for $T_{\beta\alpha}$ and μ for $\mu_{\beta\alpha}$ as their dependence on β and α is understood. If $\alpha + \beta > 2$, T has two or more discontinuities and we shall refer to such transformations as $T_n(x) = \beta x + \alpha_n \bmod 1$, $T_n(1) = \lim_{x \uparrow 1} T_n(x)$ with invariant probability μ_n , where $n < \beta + \alpha_n \leq n+1$; and we reserve the notation T, μ, α for those transformations with one discontinuity only.

The following facts are already known concerning these transformations

1) there exists a $T(T_n)$ invariant probability $\mu(\mu_n)$ with $\mu \ll 1$,

$(\mu_n \ll 1)$, given by $\mu(E) = \int_E h(x) dx$ where

$h(x) = K \sum_{n=0}^{\infty} \beta^{-n} \left(\chi_{[0, T^n(1)]}(x) - \chi_{[0, T^n(0)]}(x) \right)$ and K is a

or normalizing factor [P 1].

- 2) only one such $T(T_n)$ -invariant probability measure $\mu(\mu_n)$ exists, therefore $T(T_n)$ is ergodic with respect to $\mu(\mu_n)$ [C].
- 3) $T(T_n)$ is WB for the following cases
- (i) $\alpha = 0$, (ii) $\beta + \alpha \in \mathbb{Z}$ (iii) $\beta \geq 2$ [A.W] [W 2] [S].
- 4) a. for $\beta > 2$ $\text{spt } \mu_n = [0, 1]$ for all $n \geq 2$.
 b. for $2 > \beta > 2^{\frac{1}{2}}$
- (i) $\text{spt } \mu = [0, 1]$,
 (ii) $\text{spt } \mu_2 = [0, 1] \setminus (T_2(1), T_2(0))$ for $1/\beta \leq \alpha_2 \leq 3\beta - 1 - \beta^2/\beta$,
 (iii) $\text{spt } \mu_2 = [0, 1]$ for $\alpha_2 < 1/\beta$ or $\alpha_2 > 3\beta - 1 - \beta^2/\beta$ [C].
- 5) a. for $2^{\frac{1}{2}} < \beta < 2$ T is WB [B]
 b. for $\beta \leq 2^{\frac{1}{2}}$ and $\alpha = (2 - \beta)/2$ T is not WB; and for $2^{1/2^{n+1}} < \beta \leq 2^{1/2^n}$ T has an eigenvalue a 2^n -th-root of unity with T^{2^n} WB on each of its invariant sets. For $\beta = 2^{1/2^n}$ T is periodic Markov with period 2^n [P 2].
- 6) Hofbauer [H 1] has studied the maximal measures of T and T_2 and these he shows to be unique and equal to μ and μ_2 respectively. In [H 2], using symbolic dynamics, the regions in the (β, α) -plane for which the supports of μ and μ_2 are not maximal are determined.

In §1 we show that for a certain range of values of α_2 , $T_{\beta\alpha_2}$ is isomorphic to $T_{\beta\alpha'}$ for some α' with $\beta + \alpha' < 2$. Following similar methods to Bowen in [B], we give sufficient conditions for T and T_2 to be weak mixing, which is known to be a sufficient condition for weak Bernoullicity [B]. Using this result we prove T_2 is WB for $\beta > 2^{\frac{1}{2}}$.

In §2 we study certain types of periodic points of T and show $T(T_2)$ is WB if and only if $\text{spt } \mu(\mu_2)$ is maximal, with the exception of a countable set of periodic Markov cases;

and if T is not WB, $(\text{spt } \mu)^c$ contains a periodic n -cycle for some n . The precise regions in the (β, α) -plane in which T is WB are given.

In §3 we determine the eigenvalues of all those transformations $T(T_2)$ which are not WB.

We shall use continually the following results:

Theorem A [L.Y]

Let f be a piecewise C^2 function of $[0,1]$ such that $\inf |f'(x)| > 1$, then there is a finite collection of closed sets I_0, I_1, \dots, I_n in $[0,1]$ such that

- (i) I_j is a finite union of closed intervals $0 \leq j \leq n$
- (ii) if $k \neq j$ then $I_k \cap I_j$ is finite
- (iii) $\bigcup_{j=0}^n I_j = [0,1]$
- (iv) for $j > 0$ $fI_j = I_j$ a.e.(1) and $\text{int } I_j$ contains at least one discontinuity of f or f'
- (v) there exists f_1, f_2, \dots, f_n in $L^1[0,1]$ such that the measures $f_1 1, \dots, f_n 1$ are positive f -invariant probabilities with supports I_1, \dots, I_n respectively.

Theorem B [B]

If f satisfies the conditions of Theorem A and f is weak mixing with respect to m , $m \ll 1$, then the natural extension of f is WB.

Notes 1) All measures considered in this section will be absolutely continuous with respect to Lebesgue measure 1.

2) In some of the above results we have confused the maps T and T_n with the maps $T|_{[0,1)}$ and $T_n|_{[0,1)}$ respectively. As all measures under consideration are absolutely continuous with respect to 1, this does not affect the measure theoretic properties studied here and the results we deduce will hold for both these maps.

In the following when we write $T(x) = \beta x + \alpha \bmod 1$ it is to be understood that $T(1) = \lim_{x \uparrow 1} T(x)$.

I would like to acknowledge my indeptedness to the works of R.Cabane and F.Hofbauer for initiating this study.

$$1. \quad \beta + \alpha = \beta + \frac{\beta\alpha_2 - 1}{\beta - 1} \leq \beta - \frac{1}{\beta - 1} + \frac{\beta}{\beta - 1} \left\{ \frac{3\beta - 1 - \beta^2}{\beta} \right\} = \frac{2\beta^2 - 2\beta}{\beta^2 - \beta} = 2$$

$$\beta + \alpha_2 \geq \beta + \frac{1}{\beta} > 1 \quad \text{for } \beta > 1.$$

$$3. \quad \alpha_2 = \frac{3\beta - 1 - \beta^2}{\beta} \Rightarrow \alpha = \frac{3\beta - 1 - \beta^2 - 1}{\beta - 1} = \frac{2(\beta - 1) - \beta(\beta - 1)}{\beta - 1} = 2 - \beta.$$

$$4. \quad \alpha_2 = \frac{3 - \beta}{2} \Rightarrow \alpha = \frac{\beta(3 - \beta) - 2}{2(\beta - 1)} = \frac{2(\beta - 1) - \beta(\beta - 1)}{2(\beta - 1)} = \frac{2 - \beta}{2}$$

$$\text{If } Tx = \begin{cases} \beta x + \alpha & 0 \leq x < \frac{1 - \alpha}{\beta} \\ \beta x + \alpha - 1 & \frac{1 - \alpha}{\beta} \leq x < 1 \end{cases} \text{ then } Sx = 1 - T(1 - x) \text{ is}$$

given by $Sx = \beta x + \gamma \pmod{1}$ where $\gamma = 2 - \alpha - \beta$.

and $\alpha > \frac{2 - \beta}{2} \Rightarrow \gamma < \frac{2 - \beta}{2}.$

§1. A simple isomorphism and conditions for weak Bernoullicity.

Theorem 1. If $1/\beta \leq \alpha_2 \leq (3\beta - 1 - \beta^2)/\beta$ and

$\alpha = (\beta \alpha_2 - 1)/(\beta - 1)$ then $T(x) = \beta x + \alpha \pmod{1}$ is measure theoretically isomorphic to $T_2(x) = \beta x + \alpha_2 \pmod{1}$.

Remarks. 1. $\beta + \alpha \leq 2$ and $\beta + \alpha_2 > 2$.

2. $\text{spt } \mu_2 \subset [0, T_2(1)] \cup [T_2(0), 1], [0]$.

3. As α_2 ranges from $1/\beta$ to $(3\beta - 1 - \beta^2)/\beta$, α ranges from 0 to $2 - \beta$.

4. We need only consider the cases when $\alpha \leq (2 - \beta)/2$ and $\alpha_2 \leq (3 - \beta)/2$, for the cases when $\alpha > (2 - \beta)/2$ and $\alpha_2 > (3 - \beta)/2$ are covered by the maps $1 - T(1 - x)$ and $1 - T_2(1 - x)$, which are also linear mod 1 maps.

Proof of Theorem 1.

We consider three cases

- a) $\beta \geq (1 + 5^{\frac{1}{2}})/2$ and $1/\beta \leq \alpha_2 \leq (3\beta - 1 - \beta^2)/\beta$ or $1 < \beta < (1 + 5^{\frac{1}{2}})/2$ and $(2\beta - \beta^2 + 1)/(\beta + 1) \leq \alpha_2 \leq 2/(\beta + 1)$
- b) $1 < \beta < (1 + 5^{\frac{1}{2}})/2$ and $1/\beta \leq \alpha_2 < (2\beta - \beta^2 + 1)/(\beta + 1)$
- c) $1 < \beta < (1 + 5^{\frac{1}{2}})/2$ and $2/(\beta + 1) < \alpha_2 \leq (3\beta - 1 - \beta^2)/\beta$.

By remark 4, c) is similar to b) and we need not consider it separately.

Case a). These conditions ensure that

- (i) $T_2^2(1) \leq T_2(1)$ and $(1 - \alpha_2)/\beta \leq T_2(1)$,
- (ii) $(2 - \alpha_2)/\beta \geq T_2(0)$ and $T_2(0) \leq T_2^2(0)$ and
- (iii) $T(0) \leq (1 - \alpha)/\beta$ and $T(1) \geq (1 - \alpha)/\beta$.

See diagram 1a.

$$l(c) = \frac{\beta^2 - 2\beta - 1 + (\beta + 1)\alpha_2}{\beta - 1}.$$

Let $A_2 = (0, T_2^2(1))$ then $l(A_2) = \beta\alpha_2 + \beta^2 - 2\beta + \alpha_2 - 1$,

$B_2 = (T_2^2(1), T_2(1))$ then $l(B_2) = 3\beta - \beta^2 - \beta\alpha_2 - 1$,

$C_2 = (T_2(0), T_2^2(0))$ then $l(C_2) = \beta\alpha_2 - 1$,

$D_2 = (T_2^2(0), 1)$ then $l(D_2) = 2 - \beta\alpha_2 - \alpha_2$;

and let

$A = (0, T(0))$ then $l(A) = (\beta\alpha_2 - 1)/(\beta - 1)$,

$B = (T(0), (1 - \alpha)/\beta)$ then $l(B) = (2 - \alpha_2 - \beta\alpha_2)/(\beta - 1)$,

$C = ((1 - \alpha)/\beta, T(1))$ then $l(C) = \frac{\beta^2 - \alpha_2 - \beta\alpha_2}{\beta - 1}$, and

$D = (T(1), 1)$ then $l(D) = (3\beta - \beta^2 - \beta\alpha_2 - 1)/(\beta - 1)$.

T_2 covers twice A_2 and D_2 , and covers once B_2 and C_2 .

T covers twice B and C , and covers once A and D .

Define a map $\phi: [0, 1] \rightarrow X$, where $X = [0, 1] \setminus (T_2(1), T_2(0))$, such

that ϕ is linear, with gradient $\beta - 1$, on each of the intervals

A, B, C and D , and $\phi(A) = C_2$, $\phi(B) = D_2$, $\phi(C) = A_2$ and $\phi(D) = B_2$.

Explicitly define ϕ by

$$\phi(x) = (\beta - 1)x + \alpha_2 \in C_2, \text{ for } x \in A$$

$$\phi(x) = (\beta - 1)(x - T(0)) + T_2^2(0) \in D_2, \text{ for } x \in B$$

$$\phi(x) = (\beta - 1)(x - (1 - \alpha_2)/(\beta - 1)) \in A_2, \text{ for } x \in C$$

$$\phi(x) = (\beta - 1)(x - T(1)) + T_2^2(1) \in B_2, \text{ for } x \in D.$$

ϕ is defined a.e. μ , measurable and invertible a.e. It is easy to

check that ϕ commutes with T and T_2 , and by the uniqueness of

μ and μ_2 we have $\phi\mu = \mu_2$, giving $T \stackrel{\phi}{\sim} T_2$.

Case b). In this case

$$(i) \quad T_2(1) < (1 - \alpha_2)/\beta \Rightarrow T_2^2(1) > \alpha_2,$$

$$(ii) \quad T_2(0) < (2 - \alpha_2)/\beta \Rightarrow T_2^2(0) > T_2(0) \text{ and } T_2^2(1) > T_2^2(0) \text{ and}$$

$$(iii) \quad T(1) < (1 - \alpha)/\beta.$$

See diagram 1b.

Let $A_2 = (0, T_2(1))$, $B_2 = (T_2(0), T_2^2(0))$,

$C_2 = (T_2^2(0), T_2^2(1))$, $D_2 = (T_2^2(1), 1)$

and $A = (0, T(0))$, $B = (T(0), T(1))$,

$C = (T(1), (1-\alpha)/\beta)$, $D = ((1-\alpha)/\beta, 1)$.

T_2 is ~~1-to-1~~ ^{covers once} on A_2, B_2 , and D_2 and ~~2-to-1~~ ^{covers C_2 twice} on C_2 .

T is ~~1-to-1~~ ^{covers once} on A, C and D , and ~~2-to-1~~ ^{covers B twice} on B .

Also $l(A_2)(\beta-1) = l(D)$, $l(B_2)(\beta-1) = l(A)$,

$l(C_2)(\beta-1) = l(B)$ and $l(D_2)(\beta-1) = l(C)$.

As in case a), we can define a map $\psi: [0, 1] \rightarrow X$ which is linear and has gradient $\beta-1$ on each of the intervals A, B, C and D ,

and where $\psi(A) = B_2$, $\psi(B) = C_2$, $\psi(C) = D_2$ and $\psi(D) = A_2$.

ψ is defined a.e. μ , invertible a.e. and commutes with T and T_2 .

Thus $(T, \mu) \underset{\psi}{\simeq} (T_2, \mu_2)$. //

The following corollary follows directly from Result 5, page 2.2, and Theorem 1.

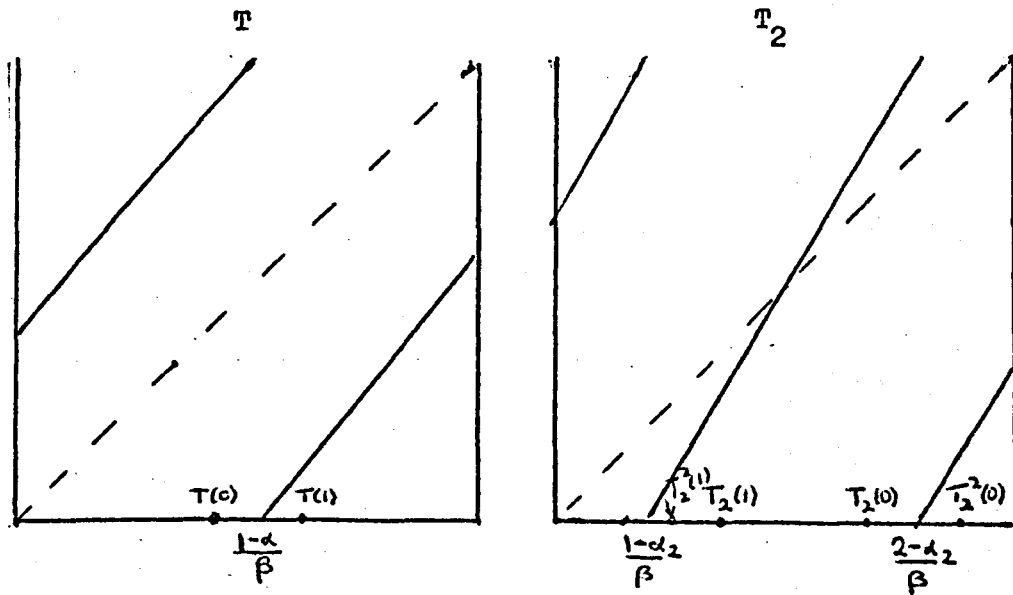
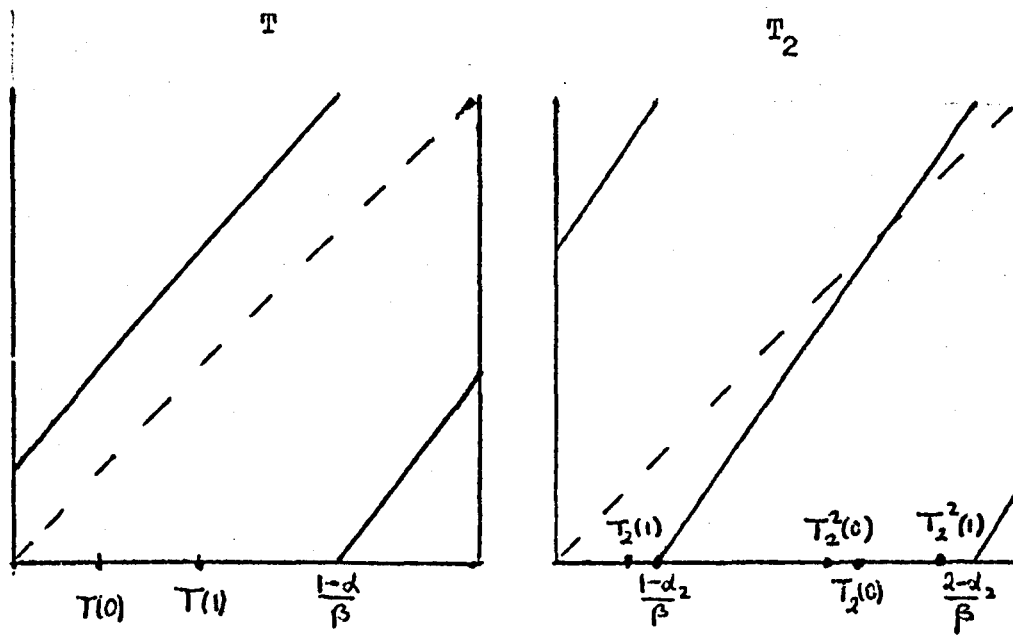
Corollary. (i) T_2 is WB for $\beta > 2^{\frac{1}{2}}$ and $1/\beta \leq \alpha_2 \leq 3\beta-1-\beta^2/\beta$.

(ii) For the symmetric transformations

$T_2(x) = \beta x + (3-\beta)/2 \bmod 1$ with $\beta \leq 2^{\frac{1}{2}}$ weak

Bernoullicity breaks down; and

$T_2(x) = 2^{1/2^n} x + (3-\beta)/2 \bmod 1$ is periodic Markov, period 2^n , with T^{2^n} WB on each of its invariant sets, $n \geq 1$.

Diagram 1a.Diagram 1b.

$$T_2(0) > \frac{1-\alpha_2}{\beta-1}$$

Conditions for weak Bernoullicity.

The following three lemmas and theorem are a simplification of those in [B], applied specifically to linear mod 1 transformations. They give sufficient conditions for T and T_2 to be weak mixing and thus WB (Theorem B).

Theorem 2. T and T_2 can only have eigenvalues which are n^{th} -roots of unity, for $n \in \mathbb{N}$.

Let \mathcal{P} be the partition $\{(a_0, a_1), (a_1, a_2)\}$ where $a_0 = 0$, $a_2 = 1$, and $a_1 = (1 - \alpha)/\beta$; and let \mathcal{P}_2 be the partition $\{(a_0, a_1), (a_1, a_2), (a_2, a_3)\}$ where $a_0 = 0, a_3 = 1, a_1 = (1 - \alpha_2)/\beta$ and $a_2 = (2 - \alpha_2)/\beta$.

In the following we shall only consider (T_2, \mathcal{P}_2) however similar (simpler) analysis gives the result for (T, \mathcal{P}) also.

Let h be the density function of μ_2 where

$h(x) = K \left(\sum_{x \in T_2^n(1)} 1/\beta^n - \sum_{x \in T_2^n(0)} 1/\beta^n \right)$ and K is a normalizing factor [P1].

Lemma 1. If $A \in \bigvee_0^k T_2^{-n} \mathcal{P}_2$, $A \neq \emptyset$ and $\bar{A} \cap \{a_0, a_1, a_2, a_3\} = \emptyset$ then $T_2(A) \in \bigvee_0^{k-1} T_2^{-n} \mathcal{P}_2$.

Proof. Let $A = \bigcap_{n=0}^k T_2^{-n}(a_{i_n}, a_{i_n+1})$ where a_{i_n} and a_{i_n+1} are elements of $\{a_0, a_1, a_2, a_3\}$.

Then $A = (a_{i_0}, a_{i_0+1}) \cap T_2^{-1} B$ where $B = \bigcap_{n=1}^k T_2^{-n+1}(a_{i_n}, a_{i_n+1})$.

To show $T_2(A) = B$, it is enough to show $B \subset T_2(a_{i_0}, a_{i_0+1})$.

T_2 maps (a_{i_0}, a_{i_0+1}) monotonically onto an interval which intersects B (as $A \neq \emptyset$).

Unless $T_2(a_{i_0}, a_{i_0+1}) \supset B$, we have $\bar{B} \cap \{T_2(a_{i_0}), T_2(a_{i_0+1})\} \neq \emptyset$

and \bar{A} contains a_{i_0} or a_{i_0+1} which is a contradiction. //

Lemma 2. Given $\gamma > 0$, for sufficiently large N there exists a set of subsets A_N of $\bigvee_{n=0}^N T_2^{-n} \mathcal{P}_2$ such that

$$(i) \quad \mu_2\left(\bigcup_{A \in A_N} A\right) > 1 - \gamma \quad \text{and}$$

$$(ii) \quad h(x)/h(y) \in [e^{-\gamma}, e^{\gamma}] \quad \text{for } x, y \text{ in } A \in A_N.$$

Proof.

$$\text{Let } h_m(x) = K \left(\sum_{\substack{n=0 \\ x < T_2^n(1)}}^m 1/\beta^n - \sum_{\substack{n=0 \\ x < T_2^n(0)}}^m 1/\beta^n \right).$$

Consider the points $\{T_2^n(1), T_2^n(0)\}_{n=1}^m$.

For large N_1 these points can be contained in atoms of $\bigvee_{n=0}^N T_2^{-n} \mathcal{P}_2$

of total μ_2 -measure less than $\gamma/2$.

$$|h(x) - h_m(x)| = K \left| \sum_{\substack{n=m+1 \\ x < T_2^n(1)}}^{\infty} 1/\beta^n - \sum_{\substack{n=m+1 \\ x < T_2^n(0)}}^{\infty} 1/\beta^n \right| \leq 2K \sum_{n=m+1}^{\infty} 1/\beta^n.$$

Given $\varepsilon > 0$, choose m so that $2K \sum_{n=m+1}^{\infty} 1/\beta^n < \varepsilon/2$. Then

choose N_1 so that if $A \in \bigvee_0^{N_1} T_2^{-n} \mathcal{P}_2$ and $A \cap \{T_2^n(1), T_2^n(0)\}_{n=1}^m = \emptyset$

we must have $|h(x) - h(y)| \leq \varepsilon$ for $x, y \in A$ since $h_m(x) = h_m(y)$ for all $x, y \in A$.

The set of all such A has measure greater than $1 - (\gamma/2)$.

Now choose $\varepsilon > 0$ so that $h(x)/h(y) \in [e^{-\varepsilon}, e^{\varepsilon}]$.

Let $N = N_1$ for this ε . //

Lemma 3. Given $\zeta > 0$, there exists M such that for $m \geq 0$ we

can find a collection of atoms $\mathcal{B} = \mathcal{B}_{m+1} \subset \bigvee_{n=0}^{m+M} T_2^{-n} \mathcal{P}_2$ with

- (i) $T_2^m B \in \bigvee_{n=0}^M T_2^{-n} \mathcal{P}_2$ for $B \in \mathcal{B}$
- (ii) $\mu_2(\tilde{B})/\mu_2(B) \in \left(\mu_2(T_2^m \tilde{B})/\mu_2(T_2^m B) \right) [e^{-\zeta}, e^{\zeta}]$ for $\tilde{B} \subset B \in \mathcal{B}$ and
- (iii) $\mu_2\left(\bigcup_{B \in \mathcal{B}} B\right) > 1 - \zeta$.

Proof. Condition (i) holds for B unless at least one of the sets

$\bar{B}, \overline{T_2 B}, \dots, \overline{T_2^{m-1} B}$ intersects $\{a_0, a_1, a_2, a_3\}$.

$T_2^k B \in \bigvee_{n=0}^{m+M-k} T_2^{-n} \mathcal{P}_2$ and so has μ_2 -measure at most $L\beta^{-(m+M-k)}$

where L is a constant (h is a bounded function).

At most 6 atoms of $\bigvee_{n=0}^{m+M-k} T_2^{-n} \mathcal{P}_2$ are adjacent to $\{a_0, a_1, a_2, a_3\}$.

Thus the total measure of all $B \in \bigvee_{n=0}^{m+M} T_2^{-n} \mathcal{P}_2$ with $\overline{T_2^k B} \cap \{a_0, a_1, a_2, a_3\} \neq \emptyset$

is at most $6L\beta^{-(M+m-k)}$.

So we can find a class \mathcal{B} of atoms of $\bigvee_{n=0}^{m+M} T_2^{-n} \mathcal{P}_2$ satisfying

condition (i) such that $\mu_2\left(\bigcup_{B \in \mathcal{B}} B\right) > 1 - (\zeta/3)$.

For $\tilde{B} \subset B \in \mathcal{B}$,

$$\mu_2(T_2^m \tilde{B}) = \int_{T_2^m \tilde{B}} h(y) dy = \int_{\tilde{B}} h(T_2^m(x)) |T_2^{m'}(x)| dx \text{ where } T_2^{m'} = \beta^n.$$

Thus $\mu_2(T_2^m \tilde{B}) = \beta^m \int_{\tilde{B}} h(T_2^m(x)) dx = \beta^m \int_{\tilde{B}} \frac{h(T_2^m(x))}{h(x)} \cdot h(x) dx$ and

$$\mu_2(T_2^m B) = \beta^m \int_B \frac{h(T_2^m(x))}{h(x)} \cdot h(x) dx \quad \text{that is,}$$

$$\mu_2(T_2^m \tilde{B}) = \beta^m \int_{\tilde{B}} \frac{h(T_2^m(x))}{h(x)} \cdot d\mu_2 \text{ and } \mu_2(T_2^m B) = \beta^m \int_B \frac{h(T_2^m(x))}{h(x)} d\mu_2.$$

By lemma 2 $h(x)$ and $h(T_2^m(x))$ will each vary on B by at most a multiplicative factor in $[e^{-\gamma/3}, e^{\gamma/3}]$ when M is large, except when B or $T_2^m B$ are in certain small sets of total measure $< 2\gamma/3$. Therefore there is a set of atoms \mathcal{B} of total measure greater than $1-\gamma$ such that $\mu_2(T_2^m \tilde{B})/\mu_2(T_2^m B) \in \mu_2(\tilde{B})/\mu_2(B) [e^{-2\gamma/3}, e^{2\gamma/3}]$, so $\mu_2(\tilde{B})/\mu_2(B) \in \mu_2(T_2^m \tilde{B})/\mu_2(T_2^m B) [e^{-\gamma}, e^{\gamma}]$ for $\tilde{B} \subset B \in \mathcal{B}$. //

Proof of Theorem 2. Let F be a bounded measurable function of $[0,1]$ and $\gamma \in \mathbb{C}$ such that $F(T_2(x)) = \gamma F(x)$ for a.a. (μ_2) x and $|\gamma| = 1$.

Let $m > 0$, $\delta_1 > 0$ and \mathcal{C}_m be the class of atoms $C \in \bigvee_{n=0}^M T_2^{-n} \mathcal{P}_2$

such that

- (i) there exists atoms $B \in \bigvee_{n=0}^{M+m} T_2^{-n} \mathcal{P}_2$ with $T_2^m B \subset C$
- (ii) $B \in \mathcal{B}_{m+M}$ of lemma 3 and
- (iii) $\mu_2(\bigcup_{B \in \mathcal{B}_{m+M}} B) > 1 - \delta_1$.

Then $\mu_2(\bigcup_{C \in \mathcal{C}_m} C) > 1 - \delta_1$. We can pick C , with $\mu_2(C) > 0$,

and $m_k \rightarrow \infty$ such that

- (i) $C \in \mathcal{C}_{m_k}$ for all k and
- (ii) given $\delta > 0$ there exists a set K such that $F|_K$ is continuous and $\mu_2(K) > 1 - (1 - \delta_1) \delta \mu_2(C)$.

For at least one atom $B \in \mathcal{B}_{m_k+M}$ with $T_2^{m_k} B = C$ we must have

$$\mu_2(B \cap K^c) \leq \delta \mu_2(B). \text{ Chose } k \text{ large enough so that for } x, y \in K$$

so that $|x-y| < \beta^{-m_k}$ implies $|F(x) - F(y)| < \delta$. Then F varies

by at most δ on $B \cap K$ so F varies by at most δ on $T_2^{m_k}(B \cap K)$

(as $|\gamma| = 1$).

Now $T_2^{m_k} B = C$ and by lemma 3

$$\mu_2(T_2^{m_k}(B \cap K^c)) \leq \mu_2(C) \mu_2(B \cap K^c) / \mu_2(B) e^{\delta_1} \leq \delta e^{\delta_1} \mu_2(C).$$

Thus F varies by at most δ on a subset of C of measure greater than $1 - \delta e^{\delta_1} \mu_2(C)$.

Now let $\delta \rightarrow 0$ and we have F constant on C .

The above holds for all C in $\mathcal{C}(\mathcal{B})$ where

$$\mathcal{C}(\mathcal{B}) = \left\{ C \in \bigvee_{n=0}^M T_2^{-n} \mathcal{P}_2 : C \in \mathcal{C}_m \text{ for infinitely many } m \right\} \text{ and}$$

we have $\mu_2(\mathcal{C}(\mathcal{B})) > 1 - 2\delta_1$.

Letting $\delta_1 \rightarrow 0$ we get a countable set of disjoint open intervals

$I_F = \{I_1, I_2, \dots\}$ such that F is μ_2 -equivalent to a constant on each I_j . Each I_j is an atom of $\bigvee_{n=0}^{M_j} T_2^{-n} \mathcal{P}_2$ for some M_j and $\mu_2(\bigcup_j I_j) = 1$.

Now F is constant on $T_2^n I_j$ a.e. μ_2 and as $\{T_2^{-n} I_j\}_{n \in \mathbb{N}}$ cannot be a disjoint set of intervals we have $\gamma^n = 1$ for some $n > 0$. //

Corollary 1. For $2^{1/n+1} < \beta \leq 2^{1/n}$ T and T_2 can only have eigenvalues N^{th} -roots of unity, where $N \leq n$.

Proof. Similar reasoning applies to T so we shall prove for T_2 only.

Let F be an eigenfunction such that $FT_2 = \gamma F$ with $\gamma^N = 1$.

We call an open interval U μ_2 -positive if $h(x) > 0$ a.e. 1 for $x \in U$.

For $r \in \mathbb{C}$, U is a maximal r -interval if

- (i) U is μ_2 -positive and open
- (ii) $F(x) = r$ a.e. on U , and
- (iii) if $V \supset U$ and V satisfies (i) and (ii) then $V = U$.

Let r be the value of F on I_1 (see Theorem 2).

Let U_1, U_2, \dots be maximal r -intervals with $1(U_1) \geq 1(U_k)$ for $k > 1$.

Define intervals W_0, W_1, \dots, W_N by

$W_0 = U_1$ and $W_{j+1} =$ longest μ_2 -positive interval contained in $T_2(W_j)$.

As $F(W_j) = r\tau^j$, these W_j are disjoint for $j < N$ and F takes

the same value on W_0 and W_N .

Note 1. $W_{j+1} = T_2(W_j)$ unless W_j contains a discontinuity of T_2 .

2. Only one W_j ($j < N$) can contain one of the discontinuities.

For suppose P_1 and P_2 are these discontinuities with

$P_1 \in W_{j_1}$ and $P_2 \in W_{j_2}$, $j_1 \neq j_2$, then $T_2 W_{j_1} \cap T_2 W_{j_2} \neq \emptyset$.

So F takes the same value on $T_2 W_{j_1} \cap T_2 W_{j_2}$ which is a contradiction.

Now suppose no W_j contains a discontinuity for $j < N$ then

$l(W_N) = \beta^N l(W_0)$ and this contradicts the maximality of $l(W_0)$.

Therefore one of the intervals contains a discontinuity for $j < N$

and we have $l(W_N) \geq \beta^N l(W_0)/2$.

Since $l(W_0) \geq l(W_N)$ it follows that $N \leq n$.

Corollary 2. T_2 is WB for $\beta > 2^{\frac{1}{2}}$.

Proof. For $1/\beta \leq \alpha_2 \leq (3\beta - 1 - \beta^2)/\beta$ we have the result by Theorem 1 and [B].

Suppose $\alpha_2 < 1/\beta$ or $\alpha_2 > (3\beta - 1 - \beta^2)/\beta$, since from corollary 1

T_2 cannot have any eigenfunctions except the constants, and by

Theorem B, T_2 is WB.

Corollary 3. Let $2^{1/n+1} < \beta \leq 2^{1/n}$ then to prove T (or T_2) is WB it is sufficient to prove T, T^2, \dots, T^n (T_2, T_2^2, \dots, T_2^n) are ergodic with respect to μ (μ_2).

§2. Periodic Points of T , the support of μ and a 'simple' condition for weak Bernoullicity.

Let $P = (1 - \alpha)/\beta$ be the point of discontinuity of $T(x) = \beta x + \alpha \pmod{1}$.

Definition 1. z is a periodic point of T , of period n , if $T^n z = z$, and $T^i z \neq z$ for $0 < i < n$. We call the set $\{T^i z\}_{i=0}^{n-1}$ an n -cycle of T .

2. $\{z_1, z_2, \dots, z_n\}$ is an $n(k)$ -cycle of T , if it is an n -cycle and $z_1 < z_2 < \dots < z_{n-k} < P < z_{n-k+1} < \dots < z_n$.

Remark. We need only consider transformations with $n(k)$ -cycles where $k \leq n/2$. For if T has an $n(k)$ -cycle where $k > n/2$ then $1 - T(1 - x)$ has an $n(n-k)$ -cycle and $n-k < n/2$.

Primary cycles. Let $\{z_i\}_{i=1}^n$ be an $n(k)$ -cycle such that

$$T(z_{n-k+i}) = z_i \text{ for } 1 \leq i \leq k \text{ and } T(z_i) = z_{k+i} \text{ for } 1 \leq i \leq n-k; \quad (2.1)$$

and let d_1 and d_2 be the smallest positive integers satisfying

$$T^{d_1-1}(z_k) = z_{n-k+1} \text{ and } T^{d_2-1}(z_{k+1}) = z_{n-k} \quad (2.2)$$

(i.e. $d_1 k = n - k + 1$ and $d_2 k + 1 = n - k$).

We call $\{z_i\}_{i=1}^n$ a primary $n(k)$ -cycle if it satisfies condition (2.1)

and $T^{d_1}(0) > P$ and $T^{d_2}(1) < P$.

A primary $n(k)$ -cycle which satisfies, further, the condition

$T(1) \leq z_{k+1}$ and $T(0) \geq z_k$ is a primary $n(k)$ -cycle of type A;

otherwise we call it a primary $n(k)$ -cycle of type B, (in this case $\{z_{k+1}, z_k\} \cap (T(0), T(1)) \neq \emptyset$).

Notes. 1. Primary $n(k)$ -cycles are characterized by the graph of T^n which has z_1, \dots, z_n as fixed points and only $n+1$ discontinuities — since $\{T^{-i}P\}$ is a one point set for $0 \leq i \leq n-2$ and a two point

set for $i = n-1$. See diagram 2.

2. An $n(k)$ -cycle can only be primary if $(n,k) = 1$,
i.e. n is coprime to k .

Secondary cycles. If $\{z_i\}_{i=1}^n$ is an $n(k)$ -cycle which satisfies

either (i) condition (2.1) and not condition (2.2)

or (ii) $(n,k) \neq 1$, ^{or any other case} (in which case either $T(z_n) > z_k$ or

$T(z_1) < z_{k+1}$, and $[T(0), T(1)]$ contains at least

two points of the cycle.)

then we call this a secondary $n(k)$ -cycle.

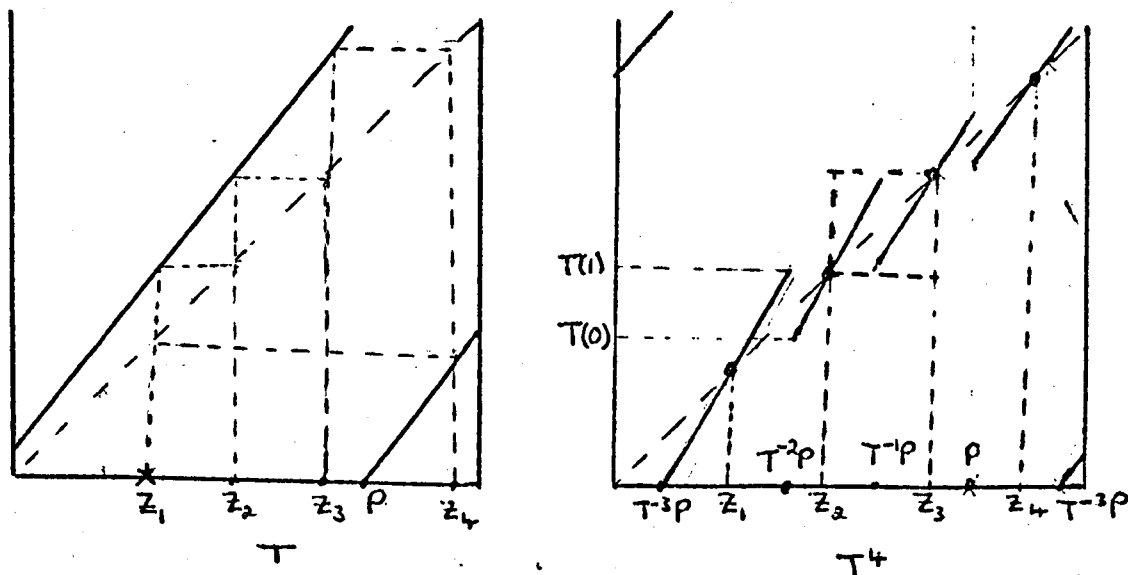


Diagram 2. Example $n = 4$, $k = 1$.

We shall now analyse these cycles to determine under what conditions they occur.

Let $[1]_n = i_0 i_1 \dots i_n$ where $i_j = 0$ if $T^j(1) \leq P$ or $i_j = 1$ if $T^j(1) > P$

$[0]_n = h_0 h_1 \dots h_n$ where $h_j = 0$ if $T^j(0) \leq P$ or $h_j = 1$ if $T^j(0) > P$

$[z_1]_n = p_0 p_1 \dots p_n$ where $p_j = 0$ if $T^j(z_1) \leq P$ or $p_j = 1$ if $T^j(z_1) > P$

$[z_n]_n = r_0 r_1 \dots r_n$ where $r_j = 0$ if $T^j(z_n) \leq P$ or $r_j = 1$ if $T^j(z_n) > P$

Lemma 1. A primary n -cycle exist if and only if,

$$\begin{aligned} i_0=1, h_0=0; \quad i_j=h_j \quad \text{for } 1 \leq j \leq n-2; \quad i_{n-1}=1, h_{n-1}=0; \quad \text{and} \\ i_n=0, h_n=1. \end{aligned} \quad (2.3)$$

Proof. From the definition of a primary cycle, if a primary cycle exists then the above conditions hold immediately.

Suppose the above conditions hold for i_j and h_j , $0 \leq j \leq n$. Then $T^{-n+2}P \in (T(0), T(1))$ as $i_{n-1} \neq h_{n-1}$ and $\{T^{-n+1}P\}$ consists of two points. Hence T^n has only $n+1$ discontinuities. As $x \uparrow T^{-i}P$, T^n takes value $T^n(x) \uparrow T^{n-i-1}(1)$ and as $x \downarrow T^{-i}P$, T^n takes value $T^n(x) \downarrow T^{n-i-1}(0)$. Our conditions imply that $T^{n-i-1}(1) > T^{-i}P$ and $T^{n-i-1}(0) \leq T^{-i}P$. Therefore between any two discontinuities of T^n we get a fixed point, giving a primary cycle of T . //

Corollary. If $\{z_i\}_{i=1}^n$ is a primary n -cycle then

$$p_0=p_n=0; \quad p_j=i_j \quad \text{for } 1 \leq j \leq n; \quad r_0=r_n=1; \quad \text{and } r_j=h_j \quad \text{for } 1 \leq j \leq n. \quad (2.4)$$

Lemma 2. Suppose T has a secondary $n(k)$ -cycle $\{z_i\}_{i=1}^n$ with

$$z_1 < z_2 < \dots < z_n \quad \text{then } T \text{ also has a primary } m\text{-cycle, where } m < n.$$

Proof. Consider separately the two types of secondary cycles

(i) If condition (2.1) is satisfied but $T^{d_1}(0) < P$ we have

$$[1]_{d_1} = 1i_1i_2\dots i_{d_1-2}10 \quad \text{and} \quad [0]_{d_1} = 0h_1h_2\dots h_{d_1-2}01 \quad \text{and}$$

by the minimality of d_1 (see (2.2)) $i_j=h_j$ for $1 \leq j \leq d_1-2$.

From lemma 1 T has a primary d_1^1 -cycle.

(If (2.1) is satisfied but $T^{d_2}(1) > P$ then T has a primary d_2^1 -cycle similarly.)

(ii) If $(n, k) \neq 1$ then $[T(0), T(1)]$ contains at least two points

of the cycle. Consider $\left\{T^i [T(0), T(1)]\right\}_{i=1}^n$.

Choose the smallest integer t such that $P \in T^{t-2}(T(0), T(1))$.

Then $i_j = h_j$ for $1 \leq j \leq t-2$; $i_0 = i_{t-1} = 1$; $h_0 = h_{t-1} = 0$ and $h_t = 1$.

By lemma 1 T has a primary t -cycle. //

Let $I_0 = \overline{(\text{spt } \mu)^c}$.

Lemma 3. If $I_0 \neq \emptyset$ then I_0 contains an m -cycle of T .

Proof. Let J_1, J_2, \dots, J_M be the maximal closed intervals in I_0 .

Consider the sequence of closed intervals $\{T^{-j}J_i\}_{j=0}^{\infty}$.

By theorem A I_0 contains only a finite number of disjoint intervals,

so $T^{-k}J_i \subset J_i$ for some $i \in \{1, 2, \dots, M\}$.

However T is onto, so this is true for all $i \in \{1, 2, \dots, M\}$.

Therefore each maximal interval in I_0 contains a periodic point of T . //

Theorem 3. T is WB if and only if $\text{spt } \mu = [0, 1]$, with the exception of a countable set of periodic Markov transformations with $\text{spt } \mu = [0, 1]$ also.

Proof. Suppose T is not WB and $2^{1/M+1} < \rho \leq 2^{1/M}$.

By the corollary to theorem 2, T has an eigenvalue an n^{th} -root of unity for $n \leq M$. Choosing the largest such n , we have sets

I_1, \dots, I_n such that $TI_i = I_{i+1}$ for $1 \leq i \leq n-1$, $TI_n = I_1$,

$P \in I_1$ and $I_0 = \overline{(I_1 \cup \dots \cup I_n)^c} = \overline{(\text{spt } \mu)^c}$.

We shall show that either $I_0 \neq \emptyset$ or T is periodic Markov.

By theorem A, $I_j = \bigcup_{i=1}^r [a_{ji}, b_{ji}]$ where $[a_{ji}, b_{ji}]$ is a

maximal interval in I_j , $1 \leq i \leq r_j < \infty$ and $1 \leq j \leq n$.

Now $T^{-1}I_j \subseteq I_{j-1} \cup I_0$ for $2 \leq j \leq n$ and $T^{-1}I_1 \subseteq I_n \cup I_0$.

Consider $T^n I_j$, either

(i) $T^n I_j \subseteq I_j$ and $T^n I_j \neq I_j$ for some $j = 1, \dots, n$, in which

case $I_0 \neq \emptyset$, or

(ii) $T^n I_j = I_j$ for all $j=1, \dots, n$ and T^n fixes each maximal interval of I_j where $j=1, 3, 4, \dots, n$ and $j \neq 2$. For if not, it permutes the intervals of some I_j and T^{nk} fixes the intervals for some $k > 1$. But this contradicts the maximality of n , so $r_j = 1$ for $j = 1, 3, 4, \dots, n$ and $r_2 = 2$. Therefore $\{T(0), T(1)\} \subseteq I_3$.

Since T^n fixes the intervals I_j , (where $[0, b_{21}] \cup [a_{22}, 1]$ is regarded as an interval), the end points of these intervals are fixed points.

By the maximality of n their period is exactly n .

Therefore $\{a_{11}, a_{31}, a_{41}, \dots, a_{n1}, a_{22}\}$ and $\{b_{11}, b_{21}, b_{31}, \dots, b_{n1}\}$ are n -cycles.

Since $(T(0), T(1)) \subseteq I_3$

$(T(0), T(1)) \cap \{a_{11}, a_{31}, \dots, a_{22}\} = \emptyset$ and \dots

$(T(0), T(1)) \cap \{b_{11}, b_{21}, \dots, b_{n1}\} = \emptyset$ both cycles are primary cycles of type A.

By note 1 page 2.15 they are the same cycle, thus $I_0 = \emptyset$.

As $T^n I_j = I_j$ for $j=1, \dots, n$, $T^n / I_j x = 2x \bmod 1$ for $j=1, 3, \dots, n$

and $T^n / I_2 x \cong 2x \bmod 1$, by the isomorphism given in Theorem 1,

that is, T is periodic Markov with period n .

Thus if T is not WB then either (i) $\text{spt } \mu \neq [0, 1]$ or

(ii) $\text{spt } \mu = [0, 1]$ and T is periodic Markov and this can

only happen when $\beta = 2^{1/n}$ for some n . There are $\phi(n)$ values

α can take in this case ($\beta = 2^{1/n}$), where ϕ is the Euler function,

one value for each primary $n(k)$ -cycle, $(n, k) = 1$.

Suppose $I_0 \neq \emptyset$. I_0 contains an m -cycle for some m (lemma 3).

Choose the smallest integer n such that I_0 contains an n -cycle.

Let $z_1 < z_2 < \dots < z_n$ be an $n(k)$ -cycle in I_0 for some k .

Suppose (i) $\{z_i\}_1^n$ is a primary cycle of type A.

We have $z_k \leq T(0) < T(1) \leq z_{k+1}$ giving

$$T^i z_k \leq T^{i+1}(0) \quad \text{and} \quad T^{i+1}(1) \leq T^i z_{k+1} \quad \text{for } 1 \leq i \leq n.$$

Therefore $[z_i, z_{i+1}]$ for $1 \leq i \leq n-1$ and

$[0, z_1] \cup [z_n, 1]$ are invariant under T^n and so T is not WB. See diagram 3.

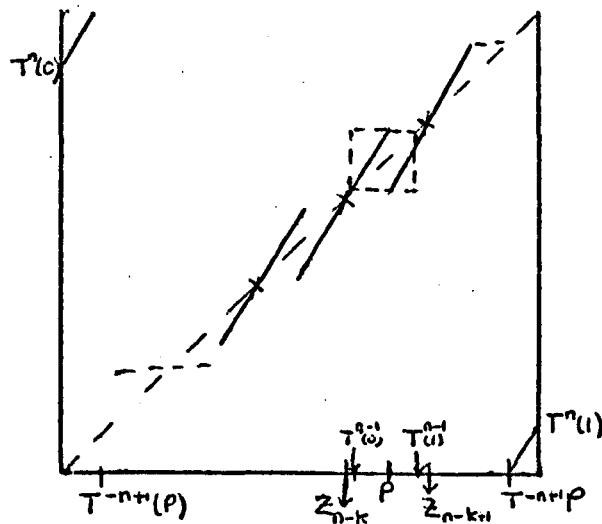


Diagram 3.

T^n

(ii) $z_1 < \dots < z_n$ is a primary cycle of type B then

$\{z_k, z_{k+1}\} \cap (T(0), T(1))$ contains one point.

Without loss in generality suppose $z_k \in (T(0), T(1))$.

Let J_i be the maximal interval in I_0 containing z_i for $1 \leq i \leq n$.

Then $J_k \subset (T(0), T(1))$, (as $\{T^j(0), T^j(1)\} \subseteq \text{spt } \mu$ for all i).

Consider $T^{-1}J_k \subseteq J_n \cup \tilde{J}_k$ where $\tilde{J}_k \subseteq (0, z_1]$ and

$$\tilde{J}_k \subseteq I_0.$$

Let W_1 be the maximal interval in I_0 containing \tilde{J}_k

and let w_1 be a periodic s -point of T contained in W_1 ,
for some $s \in \mathbb{N}$ (lemma 3).

Then $T^s W_1 \supset W_1$, $T W_1 \supset J_k$ and $T^n J_k \supset J_k$.

Whence $T^n W_1 \cap W_1 \neq \emptyset$ and by the maximality of W_1 in I_0

$W_1 \subseteq T^n W_1$. Therefore s divides n ,

Consider T^n , this has at least $s+n$ fixed points but
only $n+1$ discontinuities. Since $\beta^n > 1$ this is
impossible. Therefore $\{z_i\}_1^n$ cannot be a primary cycle
of type B.

(iii) $\{z_1, \dots, z_n\}$ is a secondary $n(k)$ -cycle.

By lemma 2 T has a primary s -cycle for $s < n$.

Let this cycle be $\{w_1, \dots, w_s\}$.

By the minimality of n , $\{w_1, \dots, w_s\} \not\subseteq I_0$ and so by (i)
and (ii) above $\{w_1, \dots, w_s\}$ must be a primary $s(k')$ -cycle
of type B for some k' . Consider T^s , this has $s+1$
discontinuities.

Without loss in generality suppose $T(1) > w_{k'+1}$.

Choose some $z_j \in [T(0), T(1)]$. See diagram 4.

Let J_j be the maximal interval in I_0 containing z_j .

Then $\{T^{-i} J_j\}_{i=0}^\infty$ contains a sequence of closed intervals

which converges either to $w_{k'}$ or to $w_{k'+1}$.

Thus either $w_{k'}$ or $w_{k'+1}$ is in I_0 , which is a contradiction.

Therefore if $I_0 \neq \emptyset$, I_0 contains a primary n -cycle of type A and
 T has an eigenvalue an n^{th} -root of unity and so is not WB. //

Corollary. If T is not WB then T has a primary n -cycle of
type A (where $\beta^n \leq 2$) and T^n supports n invariant measures
 μ^1, \dots, μ^n where $\text{spt } \mu^i = [T^i(0), T^i(1)]$ for $1 \leq i \leq n-1$ and
 $\text{spt } \mu^n = [0, T^n(1)] \cup [T^n(0), 1]$.

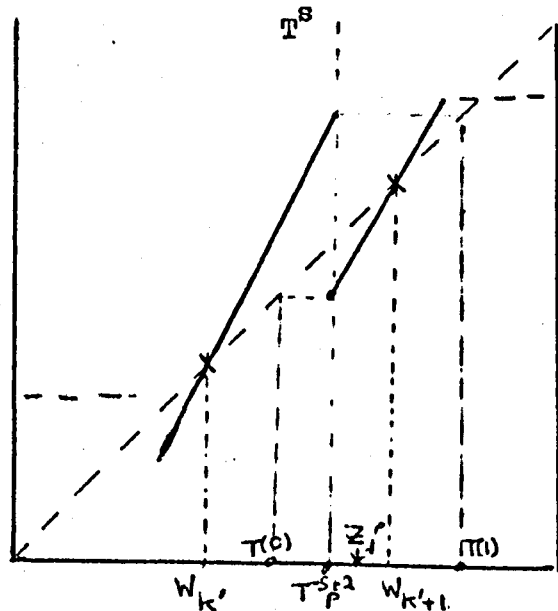
$T^n / [0, T^n(1)] \cup [T^n(0), 1]$ is isomorphic to $\beta^n x + \alpha'_n \bmod 1$ (by

Theorem 1), and

$T^n / [T^i(0), T^i(1)]$ is isomorphic to $\beta^n x + \alpha'_i \bmod 1$ for some α'_i

where $1 \leq i \leq n-1$.

Diagram 4.



To determine all the transformations $T_{\beta\alpha}$ that are not WB it is sufficient to determine all values of β and α for which primary $n(k)$ -cycles of type A exist, where $\beta \leq 2^{1/n}$ and $(n,k) = 1$.

Let $\{z_i\}_{i=1}^n$ be a primary $n(k)$ -cycle of type A, with

$k < n/2$ and $n = mk + s$ where $s < k$. Then

$$z_1 < z_2 < \dots < z_s < z_{s+1} < \dots < z_{(m-1)k+s} < P < z_{(m-1)k+s+1} < \dots < z_n$$

and by (2.3) and (2.4)

$$[1]_n = 1i_1 \dots i_{n-2} 10, \quad [0]_n = 0i_1 \dots i_{n-2} 01,$$

$$[z_1]_n = 0i_1 \dots i_{n-2} 10 \text{ and } [z_n]_n = 1i_1 \dots i_{n-2} 01.$$

We must determine the entries i_j , $1 \leq j \leq n-2$, for then we can evaluate corresponding β and α and hence the transformations $T_{\beta\alpha}$ giving these cycles.

Now $T^t z_n = z_{tk}$ for $1 \leq t \leq m$ thus $i_1 = i_2 = \dots = i_{m-1} = 0$ and $i_m = 1$.

$T^{m+t} z_n = z_{tk-s}$ for $1 \leq t \leq m-1$ thus $i_{m+1} = \dots = i_{2m-1} = 0$.

$T^{2m} z_n = z_{mk-s}$ so $i_{2m} = 0$ and $i_{2m+1} = 1$ if $k-s < s$ or

$i_{2m} = 1$ and $i_{2m+1} = 0$ if $k-s > s$.

In this way the block $i_1 i_2 \dots i_{n-2}$ is seen to be a combination of blocks of 0's of length $m-1$ and length m , separated by single 1's. There are precisely $(s-1)$ blocks of 0's of length m , $(k-s+1)$ blocks of 0's of length $(m-1)$ and $(k-1)$ occurrences of 1 separating these blocks. We must determine in which order these blocks occur.

Note. For $k=1$, $s=0$ and it is easy to see $i_j=0$ for $1 \leq j \leq n-2$. (2.5)

The following only applies for $k > 1$ (i.e. $s > 0$).

Let h_i be positive integers satisfying

$$ik = (h_1 + h_2 + \dots + h_i)s + r_i \text{ for } 0 \leq r_i < s \text{ and } 1 \leq i \leq s.$$

We have $T^{2m} z_n = z_{mk-s}$, $T^{2m+1} z_n = z_{k-2s}, \dots$, and

$T^{(h_1+1)m+1} z_n = z_{k-h_1s}$. So $i_1 i_2 \dots i_{n-2}$ begins with h_1 blocks of 0's of length $m-1$, followed by one block of 0's of length m .

Continuing in this way, we have next h_2-1 blocks of 0's of length $m-1$ followed by one block of 0's of length m

Continuing this, we finish with h_s-1 blocks of 0's of length $m-1$.

(NB. Some h_i may equal 1, in which case we have two blocks of 0's of length m , separated by 1 only).

Altogether there are $h_1 + h_2 + \dots + h_s$ blocks of 0's so $h_1 + \dots + h_s = k$, and for fixed n and k the values of s and h_i , $1 \leq i \leq s$ are

determined.

Examples 1. $n=13$, $k=3$, then $m=4$, $s=1$, and $h_1=3$.

$i_1 i_2 \dots i_{11}$ is the block 00010001000.

2. $n=13$, $k=5$, then $m=2$, $s=3$, $h_1=1$, $h_2=2$ and $h_3=2$.

$i_1 i_2 \dots i_{11}$ is the block 01001010010.

In general $i_1 i_2 \dots i_{n-2}$ is the block

$$\underbrace{0 \dots 0}_{h_1 \text{ blocks of } m-1 \text{ 0's}} \underbrace{100 \dots 010}_{\text{one block of } m \text{ 0's}} \dots \underbrace{10 \dots 0}_{h_{s-1} \text{ blocks of } m-1 \text{ 0's}} \quad (2.6)$$

We can now find expressions for $T^n(1)$, $T^n(0)$, z_1 and z_n .

Suppose $k > 1$, (see (2.5)). Define integers V_j by

$$V_j = h_1 + h_2 + \dots + h_j \quad \text{for } 1 \leq j \leq s \quad \text{and let } \alpha_m = \alpha(\beta^{m-1} + \beta^{m-2} + \dots + \beta + 1).$$

If $[1]_n = 1 i_1 i_2 \dots i_{n-2} 10$ where $i_1 \dots i_{n-2}$ is as defined in (2.6)

we have

$$T(1) = \beta + \alpha - 1,$$

$$T^2(1) = \beta^2 + \alpha\beta - \beta + \alpha = \beta^2 - \beta + \alpha_2,$$

\vdots

$$T^m(1) = \beta^m + \alpha_m - \beta^{m-1},$$

$$T^{m+1}(1) = \beta^{m+1} + \alpha_{m+1} - \beta^m - 1,$$

\vdots

$$T^{h_1 m + 1}(1) = \beta^{h_1 m + 1} + \alpha_{h_1 m + 1} - \beta^{h_1 m} - \beta^{(h_1 - 1)m} - \beta^{(h_1 - 2)m} - \dots - \beta^m - 1,$$

$$T^{h_1 m + 2}(1) = \beta^{h_1 m + 2} + \alpha_{h_1 m + 2} - \beta^{h_1 m + 1} - \beta^{(h_1 - 1)m + 1} - \beta^{(h_1 - 2)m + 1} - \dots - \beta^{m+1} - \beta,$$

\vdots

$$\begin{aligned}
T^n(1) = & \beta^n + \alpha_n - \beta^{(h_1+h_2+\dots+h_s)m+s-1} - \beta^{(h_1-1+h_2+\dots+h_s)m+s-1} - \dots \\
& - \beta^{(h_2+\dots+h_s)m+s-1} \\
& - \beta^{(h_2-1+h_3+\dots+h_s)m+s-2} - \beta^{(h_2-2+h_3+\dots+h_s)m+s-2} - \dots - \beta^{(h_3+\dots+h_s)m+s-2} \\
& \vdots \\
& - \beta^{(h_j-1+\dots+h_s)m+s-j} - \dots - \beta^{(h_{j+1}+\dots+h_s)m+s-j} \\
& \vdots \\
& - \beta^{(h_s-1)m} - \beta^{(h_s-2)m} - \dots - \beta^m - 1.
\end{aligned}$$

Substituting, we have,

$$\begin{aligned}
T^n(1) = & \beta^n + \alpha_n - \beta^{V_s m+s-1} - \beta^{(V_s-1)m+s-1} - \dots - \beta^{(V_s-V_1)m+s-1} \\
& - \beta^{(V_s-V_1-1)m+s-2} - \beta^{(V_s-V_1-2)m+s-2} - \dots - \beta^{(V_s-V_2)m+s-2} \\
& \vdots \\
& - \beta^{(V_s-V_{s-1}-1)m} - \beta^{(V_s-V_{s-1}-2)m} - \dots - \beta^m - 1.
\end{aligned}$$

If we let $W_1 = \sum_{i=0}^{V_1} \beta^{(V_s-i)m+s-1}$ and $W_j = \sum_{i=1}^{h_j} \beta^{(V_s-V_{j-1}-i)m+s-j}$ for $2 \leq j \leq s$, then

$$\underline{T^n(1) = \beta^n + \alpha_n - \sum_{j=1}^s W_j.} \quad (2.7)$$

Suppose $k = 1$ (see (2.5)), we have $i_1 = i_2 = \dots = i_{n-2} = 0$ and

$$\underline{T^n(1) = \beta^n - \beta^{n-1} - 1 + \alpha_n.} \quad (2.8)$$

Using results (2.3) and (2.4),

$$(i) \quad T(0) = \alpha = T(1) - \beta - 1,$$

$$T^2(0) = \beta\alpha + \alpha = T^2(1) - \beta^2 - \beta \text{ and it is easy to see}$$

$$\underline{T^n(0) = T^n(1) - \beta^n - \beta^{n-1} + 1.}$$

$$(ii) \quad T(z_1) = \beta z_1 + \alpha = \beta z_1 + T(0) ,$$

$$T^2(z_1) = \beta^2 z_1 + \beta \alpha + \alpha = \beta^2 z_1 + T^2(0) \quad \text{and}$$

$$T^n(z_1) = \beta^n z_1 + T^n(0) - 1 = z_1 \quad \text{therefore}$$

$$\underline{z_1 = (1 - T^n(0)) / (\beta^n - 1) .}$$

$$(iii) \quad T(z_n) = \beta z_n + \alpha - 1 = \beta z_n + T(1) - \beta ,$$

$$T^2(z_n) = \beta^2 z_n + \beta \alpha + \alpha - \beta = \beta^2 z_n + T^2(1) - \beta^2 \quad \text{and}$$

$$T^n(z_n) = \beta^n z_n + T^n(1) - \beta^n + 1 = z_n \quad \text{therefore}$$

$$\underline{z_n = (\beta^n - T^n(1) - 1) / (\beta^n - 1) .}$$

Since $\{z_1, z_2, \dots, z_n\}$ is a primary n -cycle of type A

$$T^n(1) \leq z_1 \quad \text{and} \quad T^n(0) \geq z_n , \quad \text{giving}$$

$$(1 - T^n(0)) / (\beta^n - 1) \geq T^n(1) \quad \text{and} \quad (\beta^n - T^n(1) - 1) / (\beta^n - 1) \leq T^n(0) .$$

Substituting from (i) above we deduce

$$(\beta - 1)(\beta^n - 1) / \beta \leq T^n(1) \leq (\beta - 1) / \beta . \quad (2.9)$$

Now by (2.7), if $k > 1$, $\alpha = (T^n(1) - \beta^n - \sum_{i=1}^s w_i) / (\beta^{n-1} + \beta^{n-2} + \dots + \beta + 1)$,

$$\text{and (2.9) gives} \quad \frac{\beta(\sum_{i=1}^s w_i - \beta^n - \beta + 1)}{\beta(\beta^{n-1} + \dots + \beta + 1)} \leq \alpha \leq \frac{(\beta - 1 - \beta^{n+1} + \beta \sum_{i=1}^s w_i)}{\beta(\beta^{n-1} + \dots + \beta + 1)}$$

By (2.8), if $k=1$, $\alpha = (T^n(1) + \beta^{n-1} + 1 - \beta^n) / (\beta^{n-1} + \dots + \beta + 1)$ and

and (2.9) gives

$$\frac{1}{\beta(\beta^{n-1} + \dots + \beta + 1)} \leq \alpha \leq \frac{(\beta^n - \beta^{n+1} + 2\beta - 1)}{\beta(\beta^{n-1} + \dots + \beta + 1)} .$$

Summarising the above we have proved the following ,

Theorem 4. Let $T(x) = \beta x + \alpha \pmod{1}$, $2^{1/N+1} < \beta \leq 2^{1/N}$, then T is not WB if α , or $2-\beta-\alpha$ (by remark page 2.15), is in one of the following regions for some $n \leq N$ and some k where $(n,k)=1$, $k > 1$, and $k < n/2$,

$$(i) \quad \frac{1}{\beta(\beta^{n-1} + \dots + \beta + 1)} \leq \frac{\alpha}{2-\beta-\alpha} \leq \frac{(\beta^n - \beta^{n+1} + 2\beta - 1)}{\beta(\beta^{n-1} + \dots + \beta + 1)}$$

$$(ii) \quad \frac{\beta(\sum_{i=1}^s W_i) - \beta^n - \beta + 1}{\beta(\beta^{n-1} + \dots + \beta + 1)} \leq \frac{\alpha}{2-\beta-\alpha} \leq \frac{\beta(\sum_{i=1}^s W_i) - \beta^{n+1} + \beta - 1}{\beta(\beta^{n-1} + \dots + \beta + 1)}$$

where s and m are given by $n = mk + s$, $0 < s < k$; and

$$W_1 = \sum_{i=0}^{V_1} \beta^{(V_s - i)m + s - 1} \quad \text{and} \quad W_j = \sum_{i=1}^{h_j} \beta^{(V_s - V_{j-1} - i)m + s - j} \quad \text{for } 2 \leq j \leq s$$

where V_i satisfies $ik = V_i s + r_i$ for $0 \leq r_i < s$ and $1 \leq i \leq s$, and $h_1 = V_1$, $h_p = V_p - V_{p-1}$ for $2 \leq p \leq s$.

Note. The periodic Markov case, for which $\text{spt} \mu = [0, 1]$, occurs when $\beta^n = 2$ and equality hold in either (i) or (ii) above.

See diagram 5, page 2.36.

Theorem 5. Let $T_2(x) = \beta x + \alpha_2 \pmod{1}$ where $\alpha_2 + \beta > 2$ and

suppose $\alpha_2 < 1/\beta$ or $\alpha_2 > (3\beta - \beta^2 - 1)/\beta$ then

a) $\text{spt} \mu_2 = [0, 1]$ and b) T_2 is WB.

Remark. For $1/\beta \leq \alpha_2 \leq (3\beta - \beta^2 - 1)/\beta$, by Theorems 1 and 4, $\text{spt} \mu_2 \subseteq [0, 1] \setminus (T_2(1), T_2(0))$ and we can determine the values of α_2 for which T_2 is WB.

Proof. Let $P_1 = (1 - \alpha_2)/\beta$ and $P_2 = (2 - \alpha_2)/\beta$ be the discontinuities of T_2 and let $z = (1 - \alpha_2)/(\beta - 1)$ be the fixed point of T_2 .

The conditions on α_2 imply $z \notin [T_2(1), T_2(0)]$.

Suppose $2^{1/N+1} < \beta \leq 2^{1/N}$.

By remark 3, page 2.5, we need only consider the case when $\alpha_2 < 1/\beta$, thus $z > T_2(0)$.

a) Suppose $\text{spt } \mu_2 \neq [0, 1]$.

Let $I_0 = (\text{spt } \mu_2)^c$. I_0 is a finite union of intervals.

Let J be an interval in I_0 then $T_2^{-1}J \cap [P_1, P_2] = J_1 \subseteq I_0$.

Let $J_i = T_2^{-1}J_{i-1} \cap [P_1, P_2] = [a_i, b_i]$ then $J_i \subseteq I_0$ for all $i > 1$.

Therefore $a_i \rightarrow z$ and $b_i \rightarrow z$ so $z \in I_0$.

Thus $I_0 \cap (0, P_1) \neq \emptyset$ since $T_2(0) < z$. (2.10)

Now there are maximal intervals N_0, N_1 in $\text{spt } \mu_2$ such that $0 \in N_0$ and $1 \in N_1$ (since $P_i \in \text{int}(\text{spt } \mu_2)$ for $i = 1, \text{ or } 2$).

By (2.10) $P_1 \notin N_0$.

Consider $T_2(0) \in [P_1, z]$. $T_2^i(0) \in [P_1, z]$ for $i < n_0$, for some $n_0 > 0$.

We shall calculate the minimum value for n_0 such that $T_2^{n_0}(0) \in (0, P_1)$.

See diagram 6.

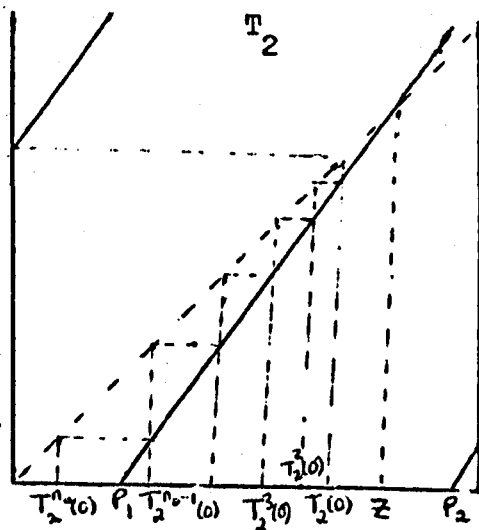


Diagram 6.

$$T_2(0) = \alpha_2,$$

$$T_2^2(0) = \beta \alpha_2 + \alpha_2 - 1,$$

.

.

.

$$T_2^{n_0-1}(0) = \alpha_2(\beta^{n_0-2} + \dots + \beta + 1) - (\beta^{n_0-3} + \dots + \beta + 1).$$

If $T_2^{n_0}(0) < P_1$ then $(\beta^{n_0-1} + \dots + \beta + 1)\alpha_2 - (\beta^{n_0-2} + \dots + 1) < (1 - \alpha_2)/\beta$
giving $\alpha_2 < (\beta^{n_0} - 1)/(\beta^{n_0+1} - 1)$.

Since $2 - \beta < \alpha_2 < 1/\beta$ we have $(2 - \beta)(\beta^{n_0+1} - 1) < \beta^{n_0} - 1$.

Thus $n_0 \in \mathbb{Z}^+$ must satisfy $\beta^{n_0} - \beta^{n_0+1} + 1 < 0$. (2.11)

Let $l(N_0) = d$ and $l(N_1) = f$ (where l is Lebesgue measure)
and let m_0 and m_1 be the smallest integers such that

$$T_2^{m_0}(N_0) \cap \{P_1, P_2\} \neq \emptyset \quad \text{and} \quad T_2^{m_1}(N_1) \cap \{P_1, P_2\} \neq \emptyset.$$

By the maximality of N_0 and N_1 the longest interval containing
 P_1 or P_2 in $\text{spt } \mu_2$ has length $\leq (d+f)/\beta$ and we have

$$l(T_2^{m_1}(N_1)) \leq (d+f)/\beta \quad \text{and} \quad l(T_2^{m_0}(N_0)) \leq (d+f)/\beta.$$

Thus $\beta^{m_0+1} d \leq d+f$ and $\beta^{m_1+1} f \leq d+f$ giving

$$(\beta^{m_0+1} - 1)d \leq f \quad \text{and} \quad (\beta^{m_1+1} - 1)f \leq d.$$

$$\text{Whence } (\beta^{m_0+1} - 1)(\beta^{m_1+1} - 1) \leq 1. \quad (2.12)$$

Now $m_0 \geq n_0$, so by (2.11) $\beta^{m_0+1} - \beta^{n_0} - 1 > 0$

giving $\beta^{m_0} > 1/(\beta - 1)$. With this and (2.12),

$(\beta^{m_1+1} - 1)/(\beta - 1) < \beta^{m_1}$ from which we deduce $\beta^{m_1} < 1$, an
impossibility.

Thus $\text{spt } \mu_2 = [0, 1]$.

b) Suppose T_2 is not WB. Then T_2 has an eigenvalue an n^{th} -root
of unity, for some n , and T_2^n has n invariant sets I_1, \dots, I_n

where each I_j is a finite union of closed intervals,

$$T_2^j I_1 = I_{j+1} \quad \text{for } 1 \leq j \leq n-1 \quad \text{and} \quad \bigcup_{j=0}^{n-1} I_j = [0, 1].$$

By a) $z \in I_k$ for some k .

Since z is a fixed point of T_2 the above decomposition is
impossible. Hence T_2 is WB. //

§3. The Eigenvalues of T.

$$\text{Let } \tilde{Y}_1(\beta; n) = \left\{ \alpha : \alpha = x \text{ or } \alpha = 2 - \beta - x \text{ where } \frac{1}{\beta(\beta^{n-1} + \dots + 1)} \leq x \leq \frac{\beta^n - \beta^{n+1} + 2\beta - 1}{\beta(\beta^{n-1} + \dots + \beta + 1)} \right\}$$

For $k > 1$, $(n, k) = 1$ and $k < n/2$ let

$$\tilde{Y}_k(\beta; n) = \left\{ \alpha : \alpha = x \text{ or } \alpha = 2 - \beta - x \text{ where } \frac{\beta(\sum_{j=1}^s W_j^k(\beta)) - \beta^n - \beta + 1}{\beta(\beta^{n-1} + \dots + \beta + 1)} \leq x \leq \frac{\beta(\sum_{j=1}^s W_j^k(\beta)) - \beta^{n+1} + \beta - 1}{\beta(\beta^{n-1} + \dots + \beta + 1)} \right\}$$

where $n = mk + s$, $0 < s < k$;

V_i satisfies $ik = V_i s + r_i$, $0 \leq r_i < s$, $1 \leq i \leq s$;

$h_1 = V_1$ and $h_p = V_p - V_{p-1}$ $2 \leq p \leq s$;

$W_1^k(\beta) = \sum_{i=0}^{V_1} \beta^{(V_s - i)m + s - 1}$ and $W_j^k(\beta) = \sum_{i=1}^{h_j} \beta^{(V_s - V_{j-1} - i)m + s - j}$ $2 \leq j \leq s$.

$$\text{Now let } \tilde{Y}(\beta; n) = \tilde{Y}_1(\beta; n) \cup \bigcup_{\substack{k, k < n/2 \\ (n, k) = 1}} \tilde{Y}_k(\beta; n) .$$

We wish to determine all the eigenvalues of the transformation T .

In Theorem 4 we showed that if $\alpha \in \tilde{Y}(\beta; n)$ then T is not WB and has an eigenvalue an n^{th} -root of unity.

Suppose $n = p_1$, p_1 a prime, then T has an eigenvalue a p_1^{th} -root of unity if and only if $\alpha \in \tilde{Y}(\beta; p_1)$ as secondary p_1 -cycles cannot exist and so $\alpha \notin \tilde{Y}(\beta; t)$ for any $t < p_1$.

Let $\{z_1, z_2, \dots, z_{p_1}\}$ be a primary $p_1(k)$ -cycle of type A then

$[z_{p_1-k}, z_{p_1-k+1}]$ is invariant under T^{p_1} and

supports a T^{p_1} -invariant measure on $[T^{p_1-1}(0), T^{p_1-1}(1)]$.

$P \in [T^{p_1-1}(0), T^{p_1-1}(1)]$ (where $P = (1 - \alpha)/\beta$) .

Now $T^{p_1} \left[\begin{matrix} T^{p_1-1}(0), T^{p_1-1}(1) \end{matrix} \right] (x) = \beta^{p_1} x + \alpha[1] \text{ (after magnification)}$

where $\alpha[1] = T^{2p_1-1}(0) \cdot (T^{p_1-1}(1) - T^{p_1-1}(0))$.

We shall evaluate $\alpha[1]$ in terms of α .

Now $T^{p_1}(0) = T^{p_1}(1) - \beta^{p_1} - \beta^{p_1-1} + 1$, that is

$$= \alpha(\beta^{p_1-1} + \dots + \beta + 1) - \beta^{p_1-1} + 1 - \sum_{j=1}^s w_j^k(\beta) \text{ when } \alpha \in \tilde{Y}_k(\beta; p_1)$$

and $k > 1$. As $z_{p_1} \leq T^{p_1}(0) < 1$,

$$\begin{aligned} T^{2p_1-1}(0) &= T^{p_1-1}(T^{p_1}(0)) = \beta^{p_1-1}(T^{p_1}(0)) + \alpha(\beta^{p_1-1} + \dots + 1) - \sum_{j=1}^s w_j^k(\beta) \\ &= \alpha(\beta^{2p_1-2} + \beta^{2p_1-3} + \dots + \beta + 1) - \beta^{2p_1-1} + \beta^{p_1-1} - (\beta^{p_1-1} + 1) \sum_{j=1}^s w_j^k(\beta) \end{aligned}$$

and $T^{p_1-1}(1) - T^{p_1-1}(0) = \beta^{p_1} + \beta^{p_1-1} - 1$.

Therefore when $\alpha \in \tilde{Y}_k(\beta; p_1)$ for $k > 1$,

$$\alpha[1] = (\beta^{p_1} + \beta^{p_1-1} - 1) \left[\alpha(\beta^{2p_1-2} + \dots + 1) - \beta^{2p_1-1} + \beta^{p_1-1} - (\beta^{p_1-1} + 1) \sum_{j=1}^s w_j^k(\beta) \right]$$

and when $\alpha \in \tilde{Y}_1(\beta; p_1)$

$$\alpha[1] = (\beta^{p_1} + \beta^{p_1-1} - 1) \left[\alpha(\beta^{2p_1-2} + \dots + \beta + 1) - 2\beta^{2p_1-2} - \beta^{p_1-1} - 1 \right].$$

Again using Theorem 4 we can determine the eigenvalues of

$\beta^{p_1} x + \alpha[1] \bmod 1$. Suppose T^{p_1} has an eigenvalue a p_2^{th} -root of unity

(on each of its invariant sets), that is, T^{p_1} has a primary p_2 -cycle

(in each invariant set) then $\alpha[1] \in \tilde{Y}(\beta^{p_1}; p_2)$ and T has an

eigenvalue a $p_1 p_2^{\text{th}}$ -root of unity.

T could also have an eigenvalue a $p_1 p_2^{\text{th}}$ -root of unity if

either (i) it has a primary p_2 -cycle of type A and T^{p_2} has a

primary p_1 -cycle of type A (on each invariant set), i.e.

$$\alpha \in \tilde{Y}(\beta; p_2) \text{ and } \alpha[1] \in \tilde{Y}(\beta^{p_2}; p_1)$$

or (ii) it has a primary $p_1 p_2$ -cycle of type A, i.e. $\alpha \in \tilde{Y}(\beta; p_1 p_2)$.

These are the only cases when T has a $p_1 p_2^{\text{th}}$ -root of unity.

Generalizing the above, we can determine for which values of α T will have an eigenvalue an N^{th} -root of unity, where

$N = p_1 p_2 \cdots p_n$. We do this by induction on n .

Define the following sets

$$Y_1(\beta; p_1) = \left\{ \alpha; \alpha \in \tilde{Y}_1(\beta; p_1) \text{ and let} \right. \\ \left. \alpha[1] = (\beta^{p_1} + \beta^{p_1-1} - 1) \left[\alpha (\beta^{2p_1-1} + \dots + 1) - 2\beta^{2p_1-2} - \beta^{p_1-1} - 1 \right] \right\}$$

$$Y_k(\beta; p_1) = \left\{ \alpha; \alpha \in \tilde{Y}_k(\beta; p_1) \text{ and let} \right. \\ \left. \alpha[1] = (\beta^{p_1} + \beta^{p_1-1} - 1) \left[\alpha (\beta^{2p_1-2} + \dots + 1) - \beta^{2p_1-1} + \beta^{p_1-1} - (\beta^{p_1-1} + 1) \sum_{j=1}^s w_j^k(\beta) \right] \right\}$$

$$Y(\beta; p_1) = Y_1(\beta; p_1) \cup \bigcup_{\substack{k, k < p_1/2 \\ (p_1, k)=1}} Y_k(\beta; p_1).$$

More generally, for any $m \geq 1$ let

$$Y_1(\beta; p_1 p_2 \cdots p_m) = \left\{ \alpha; \alpha \in \tilde{Y}_1(\beta; M) \text{ where } M = p_1 p_2 \cdots p_m, \text{ and let} \right. \\ \left. \alpha[m] = (\beta^M + \beta^{M-1} - 1) \left[\alpha (\beta^{2M-1} + \dots + 1) - 2\beta^{2M-2} - \beta^{M-1} - 1 \right] \right\}$$

For $1 < k < (p_1 p_2 \cdots p_m)/2$ and $(p_1 p_2 \cdots p_m, k) = 1$ let

$$Y_k(\beta; p_1 p_2 \cdots p_m) = \left\{ \alpha; \alpha \in \tilde{Y}_k(\beta; M) \text{ where } M = p_1 p_2 \cdots p_m, \text{ and let} \right. \\ \left. \alpha[m] = (\beta^M + \beta^{M-1} - 1) \left[\alpha (\beta^{2M-2} + \dots + 1) - \beta^{2M-1} + \beta^{M-1} - (\beta^{M-1} + 1) \sum_{j=1}^s w_j^k(\beta) \right] \right\}$$

$$Y(\beta; p_1 p_2 \cdots p_m) = Y_1(\beta; p_1 p_2 \cdots p_m) \cup \bigcup_k Y_k(\beta; p_1 p_2 \cdots p_m).$$

Generalizing still further, for $m' \geq 1$, $n \geq 1$, $\alpha[0] = \alpha$, and p_q

primes for $q = i_1, i_2, \dots, i_n, j_1, \dots, j_m$, let

$$Y_1(\beta^{p_{i_1} \dots p_{i_n}}; p_{j_1} \dots p_{j_m}) = \left\{ \alpha[n]: \alpha[n] = x \text{ or } \alpha[n] = 2 - \beta^L - x \text{ where} \right.$$

$$L = p_{i_1} \dots p_{i_n}, M = p_{j_1} \dots p_{j_m}, \text{ and}$$

$$\frac{1}{\beta^L (\beta^{L(M-1)} + \beta^{L(M-2)} + \dots + 1)} \leq x \leq \frac{\beta^{LM} - \beta^{L(M-1)} + 2\beta^L - 1}{\beta^L (\beta^{L(M-1)} + \dots + \beta^L + 1)} ;$$

$$\text{and let } \alpha[n][m'] = \alpha[n+m'] =$$

$$(\beta^{LM} + \beta^{L(M-1)} - 1) \left[\alpha[n] (\beta^{L(2M-1)} + \beta^{L(2M-2)} + \dots + \beta^L + 1) - 2\beta^{L(2M-2)} - \beta^{L-1} - 1 \right] \}$$

$$\text{and for } (k, p_{j_1} \dots p_{j_m}) = 1 \text{ and } 1 < k < (p_{j_1} \dots p_{j_m})/2 \text{ let}$$

$$Y_k(\beta^{p_{i_1} \dots p_{i_n}}; p_{j_1} \dots p_{j_m}) = \left\{ \alpha[n]: \alpha[n] = x \text{ or } \alpha[n] = 2 - \beta^L - x \text{ where} \right.$$

$$L = p_{i_1} \dots p_{i_n}, M = p_{j_1} \dots p_{j_m}, \text{ and}$$

$$\frac{\beta^L (\sum_{j=1}^s w_j^k(\beta^L) - \beta^{LM} - \beta^L + 1)}{\beta^L (\beta^{L(M-1)} + \dots + \beta^L + 1)} \leq x \leq \frac{\beta^L (\sum_{j=1}^s w_j^k(\beta^L) - \beta^{L(M+1)} + \beta^L - 1)}{\beta^L (\beta^{L(M-1)} + \beta^{L(M-2)} + \dots + \beta^L + 1)}$$

$$\text{Where } M = mk + s \text{ for } 0 < s < k ;$$

$$V_i \text{ satisfies } ik = V_i s + r_i \text{ for } 0 \leq r_i < s \text{ and } 1 \leq i \leq s;$$

$$h_1 = V_1 \text{ and } h_p = V_p - V_{p-1} \text{ for } 2 \leq p \leq s ;$$

$$w_1^k(\beta^L) = \sum_{i=0}^{V_1} \beta^{L[(V_s - i)m + s - 1]}$$

$$w_j^k(\beta^L) = \sum_{i=1}^{h_j} \beta^{L[(V_s - V_{j-1} - i)m + s - j]} \text{ for } 2 \leq j \leq s ;$$

$$\text{and let } \alpha[n][m'] = \alpha[n+m'] =$$

$$(\beta^{LM} + \beta^{L(M-1)} - 1) \left[\alpha[n] (\beta^{L(2M-2)} + \dots + \beta^L + 1) - \beta^{L(2M-1)} + \beta^{L(M-1)} - (\beta^{L(M-1)} + 1) \sum_{j=1}^s w_j^k(\beta^L) \right] \}$$

and

$$Y(\beta^{p_{i_1} \dots p_{i_n}}; p_{j_1} \dots p_{j_m}) = Y_1(\beta^{p_{i_1} \dots p_{i_n}}; p_{j_1} \dots p_{j_m}) \cup \bigcup_k Y_k(\beta^{p_{i_1} \dots p_{i_n}}; p_{j_1} \dots p_{j_m})$$

We have shown that T has an eigenvalue an N^{th} -root of unity for $N = p_1 p_2$ if

$$\alpha \in Y(\beta; p_1, p_2) = \left\{ \alpha : \alpha \in Y(\beta; p_1) \text{ and } \underline{\alpha[1]} \in Y(\beta^{p_1}; p_2) \text{ where } \alpha[1] \text{ is defined in } Y(\beta; p_1) \right\}$$

$$\cup \left\{ \alpha : \alpha \in Y(\beta; p_2) \text{ and } \underline{\alpha[1]} \in Y(\beta^{p_2}; p_1) \text{ where } \alpha[1] \text{ is defined in } Y(\beta; p_2) \right\}$$

$$\cup \left\{ \alpha : \alpha \in Y(\beta; p_1 p_2) \right\}$$

$$= \bigcup_{\sigma \in S_2} \left[\left\{ \alpha : \alpha \in Y(\beta; p_{\sigma(1)}) \text{ and } \underline{\alpha[1]} \in Y(\beta^{p_{\sigma(1)}}; p_{\sigma(2)}) \right\} \right]$$

$$\cup \left\{ \alpha : \alpha \in Y(\beta; p_1 p_2) \right\}$$

where S_2 is the permutation group on two elements.

In general if $\alpha \in Y(\beta; p_{i_1} \dots p_{i_n})$ and $\alpha[n] \in Y(\beta^{p_{i_1} \dots p_{i_n}}; p_{j_1} \dots p_{j_m})$

where $\alpha[n]$ is a value determined in $Y(\beta; p_{i_1} \dots p_{i_n})$ then T

will have an eigenvalue a $p_{i_1} \dots p_{i_n}^{\text{th}}$ -root of unity and $T^{p_{i_1} \dots p_{i_n}}$

will have an eigenvalue a $p_{j_1} \dots p_{j_m}^{\text{th}}$ -root of unity, that is, T

will have an eigenvalue a $p_{i_1} \dots p_{i_n} p_{j_1} \dots p_{j_m}^{\text{th}}$ -root of unity.

Thus we have the following theorem,

Theorem 6. Let $N = p_1 p_2 \dots p_n$, where p_i is prime for $1 \leq i \leq n$.

Then $T(x) = \beta x + \alpha \pmod{1}$ has an eigenvalue an N^{th} -root of unity

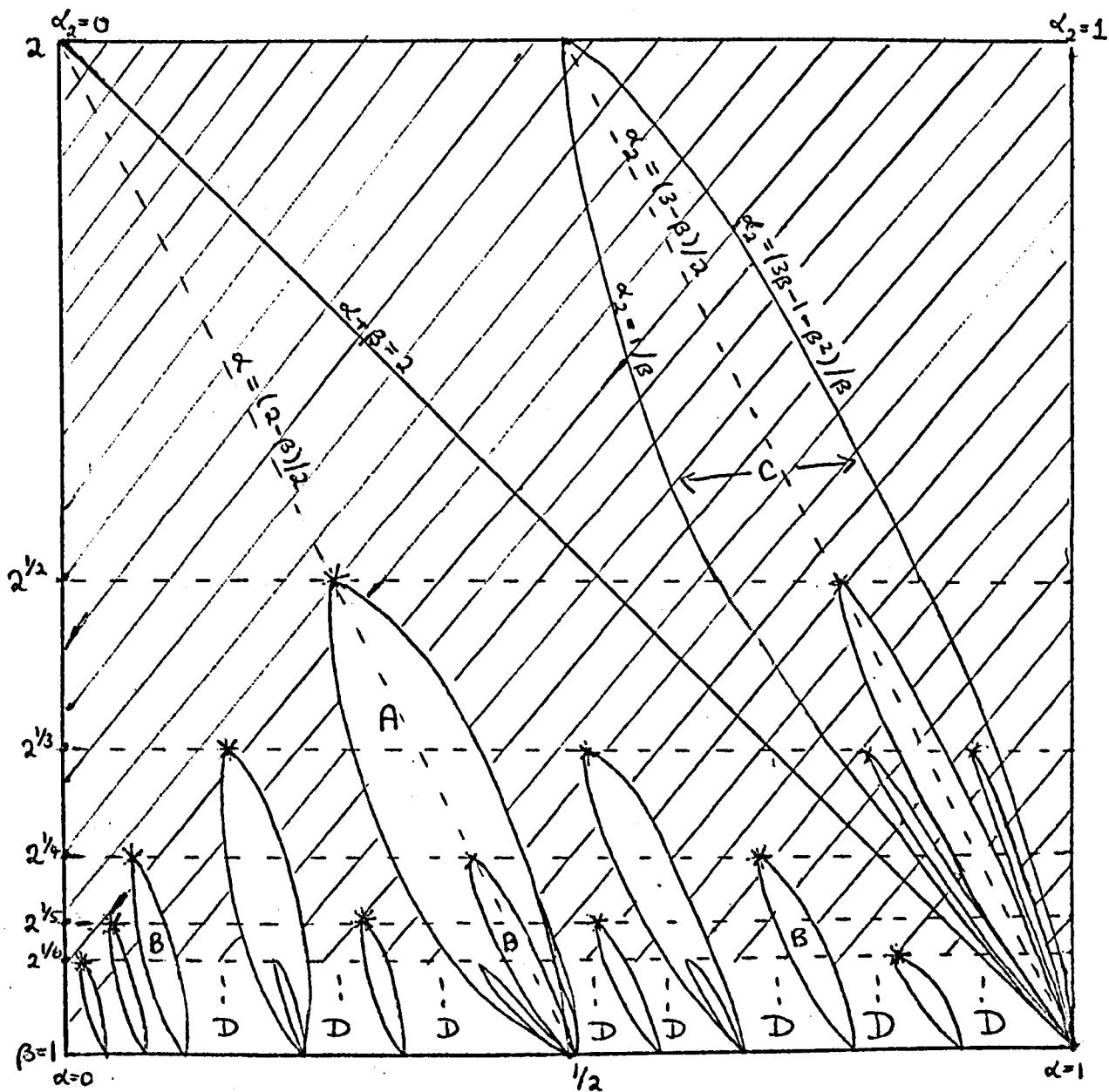
if and only if $\beta^N \leq 2$ and $\alpha \in Y(\beta; p_1, p_2, \dots, p_n)$ where

$$Y(\beta; p_1, \dots, p_n) = \bigcup_{\sigma \in S_n} \left[\bigcup_{i=0}^{n-1} \left\{ \left\{ \alpha : \alpha \in Y(\beta; p_{\sigma(1)}, \dots, p_{\sigma(i)}) \text{ and } \right. \right. \right. \\
\left. \left. \left. \alpha[i] \in Y(\beta^{p_{\sigma(1)} \dots p_{\sigma(i)}}; p_{\sigma(i+1)}, \dots, p_{\sigma(n)}) \right. \right. \right. \\
\left. \left. \left. \text{where } \alpha[i] \text{ is defined in } \right. \right. \right. \\
\left. \left. \left. Y(\beta; p_{\sigma(1)}, \dots, p_{\sigma(i)}) \right\} \right\} \right]$$

$$\bigcup \left\{ \alpha : \alpha \in Y(\beta; p_1 p_2 \dots p_n) \right\} .$$

Diagram 5 , showing the spectral properties of T and T_2 for

$$1 \leq \beta \leq 2, \quad 0 \leq \alpha \leq 1, \quad 0 \leq \alpha_2 \leq 1.$$



In region A, T has eigenvalue $2^{\frac{1}{2}}$ -root of unity.

In region B, T has eigenvalue $2^{\frac{1}{4}}$ -root of unity.

In region C, T_2 is isomorphic to some T .

In region D, the pattern continues, dividing the region further.

Shaded regions are WB, including the lines $\alpha=0$, $\alpha=1$ and $\beta=2$.

The critical points *, are the periodic Markov cases, for which

$$\text{spt } \mu = [0, 1] .$$

SECTION 3.

LINEAR MODELS FOR PIECEWISE CONTINUOUS MAPS OF THE INTERVAL.

§0. Introduction.

We consider a class \mathcal{C} of piecewise continuous and monotone increasing transformations of the unit interval I , that is, maps which display a certain global expanding or chaotic behaviour. All transformations in \mathcal{C} are shown to be topologically semiconjugate to some uniformly piecewise linear map of I and further conditions are determined for a conjugacy to exist.

These results extend those of Parry [P2] and Rand [R], where strongly transitive transformations and symmetric Poincaré maps for the Lorenz attractor, respectively are considered. Milnor and Thurston have also studied this problem of linear models for certain continuous transformations of I [M.T]. Indeed, to prove our result we use methods similar to those developed in [M.T]. Although these techniques for constructing the semiconjugacies and conjugacies here are seemingly quite different to those in [P2] there is a correspondence between the two, and when considering strongly transitive transformations the conjugacies are in fact the same.

Using these semiconjugacies we can determine the existence of certain invariant measures for transformations in \mathcal{C} . In particular, there is a unique maximal measure for f -expansions in \mathcal{C} with strong ergodic properties and compatible with Lebesgue measure l but not in general absolutely continuous with respect to l .

§1. Definitions and the class \mathcal{C} .

Let $I = [0, 1]$.

A map $g: I \rightarrow I$ is piecewise continuous and monotone increasing if

there are points P_i , $0 \leq i \leq m$, with $0 = P_0 < P_1 < \dots < P_{m-1} < P_m = 1$

such that $g|_{(P_{i-1}, P_i)}$ is continuous and strictly monotone increasing

for $1 \leq i \leq m$.

The points P_1, P_2, \dots, P_{m-1} are the points of discontinuity of g

and we call (P_{i-1}, P_i) a maximal interval of g , $1 \leq i \leq m$.

In general we say (a, b) is a maximal interval of g^n if $g^n|_{(a, b)}$

is continuous and $g^n|_{(a', b')}$ is not continuous, for any interval

(a', b') strictly containing (a, b) .

The set $\{P_1, \dots, P_{m-1}\}$ separates points if given $x, y \in I$, $x < y$,

then $g^n((x, y)) \cap \{P_1, \dots, P_{m-1}\} \neq \emptyset$ for some $n \in \mathbb{N}$. (Equivalently,

the smallest g -invariant σ -algebra containing the intervals (P_{i-1}, P_i)

is \mathcal{B} , the Borel σ -algebra of I .)

Let l_n be the number of maximal intervals of g^n and let $\beta =$

$\limsup_n \sqrt[n]{l_n}$, (written $\beta(g)$ if we need to stress its dependence on g).

Thus β is the approximate rate of production of maximal intervals

of g^n when n is large and as such is a crude measure of expansion

for g . Call β the growth number of g . We shall only consider

functions with growth number greater than 1.

The class \mathcal{C} is the set of all piecewise continuous and monotone increasing functions of I whose points of discontinuity separate points and with growth number greater than 1.

In particular, the class \mathcal{C} includes

(i) piecewise monotone increasing and expanding maps with growth

number greater than 1, for these transformations separate points.

(ii) strongly transitive piecewise continuous and monotone increasing

transformations $[P2]$.

§2. Decomposition of $g \in \mathcal{C}$.

Let $g \in \mathcal{C}$ and P_1, \dots, P_{m-1} be its points of discontinuity.

The following decomposition of I into g -irreducible sets is a topological version of that considered by Li and Yorke [L.Y] and Wagner [W]. (In [L.Y] and [W] g in addition has to satisfy certain differentiability conditions.)

Theorem 1. (c.f. Theorem ^A~~B~~, section 2.)

Let $g \in \mathcal{C}$ with discontinuities P_1, \dots, P_{m-1} then $I = \bigcup_{i=1}^N I_i$,

$1 \leq N \leq m-1$, where

- (i) I_i is a closed set for $1 \leq i \leq N$,
- (ii) $I_i \cap I_j$ is a finite set of points for $i \neq j$,
- (iii) I_i is a finite union of closed intervals,
- (iv) I_i is essentially invariant, that is ,
 $g(\text{int } I_i) \subset I_i$ and $g^{-1}(\text{int } I_i) \subset I_i$,
- (v) I_i is irreducible, in the sense that it contains no proper essentially invariant subset, and
- (vi) $\text{int } I_i$ contains at least one discontinuity of g .

Before proving this theorem, we define certain natural candidates for these g -irreducible sets and prove some elementary lemmas concerning these.

Let $\mathcal{P}_i = \bigcup_{n=0}^{\infty} g^{-n} P_i$ for $i = 1, \dots, m-1$.

Since $\{P_1, \dots, P_{m-1}\}$ separate points of I , $\bigcup_{i=1}^{m-1} \overline{\mathcal{P}_i} = I$. (3.1)

A subclass of these sets $\overline{\mathcal{P}_i}$ will be shown to be the g -irreducible sets required in Theorem 1.

Consider the following three exhaustive and exclusive possibilities for each set $\overline{\mathcal{P}_i}$ and its associated discontinuity P_i ,
 $i = 1, \dots, m-1$,

Lemma 1. $P_i \in \text{int } \overline{\mathcal{D}_i}$ for $i \in H_1$.

Proof. Let z be an interior point of $\overline{\mathcal{D}_i}$, with open neighbourhood $U \subset \overline{\mathcal{D}_i}$ satisfying, for some $M \in \mathbb{Z}^+$,

- (i) $g^M U \cap P_i \neq \emptyset$ and
- (ii) $\bigcup_{n=1}^{M-1} g^n U$ does not contain any discontinuity of g .

Such a neighbourhood exists since $i \in H_1$.

$g^M U$ is open.

Since $\mathcal{D}_i \cap U$ is dense in U , $\mathcal{D}_i \cap g^M U$ is dense in $g^M U$.

So $g^M U \subseteq \overline{\mathcal{D}_i}$ and from (i) we have $P_i \in \text{int } \overline{\mathcal{D}_i}$ //

Lemma 2. $\overline{\mathcal{D}_i}$ is essentially invariant for $i \in H_1 \cup H_2$, that is,

$$g \overline{\mathcal{D}_i} = \overline{\mathcal{D}_i} \cup F_1, \quad g(\text{int } \overline{\mathcal{D}_i}) \subseteq \overline{\mathcal{D}_i} \quad \text{and} \quad g^{-1} \overline{\mathcal{D}_i} = \overline{\mathcal{D}_i} \cup F_2,$$

$g^{-1}(\text{int } \overline{\mathcal{D}_i}) \subseteq \overline{\mathcal{D}_i}$, where F_1 and F_2 are finite, possibly empty, sets.

Proof. Let $x \in \overline{\mathcal{D}_i}$ and $U_\varepsilon(x)$ be an ε -neighbourhood of x .

For all $\varepsilon > 0$ $U_\varepsilon(x) \cap \mathcal{D}_i \neq \emptyset$.

Therefore $U_\varepsilon(gx) \cap \mathcal{D}_i \neq \emptyset$ unless x is a boundary point of $\overline{\mathcal{D}_i}$ and a point of discontinuity, (that is, unless x is a right (left) hand limit of points in $\overline{\mathcal{D}_i}$ and g is continuous from the left (right) at x).

Giving $g(\text{int } \overline{\mathcal{D}_i}) \subseteq \overline{\mathcal{D}_i}$ and $g(\overline{\mathcal{D}_i}) \subseteq \overline{\mathcal{D}_i} \cup F_1$ where F_1 is at most an $(m-1)$ point set.

Let $y \in \{g^{-1}x\}$.

$U_\varepsilon(y) \cap \mathcal{D}_i \neq \emptyset$, for all $\varepsilon > 0$, unless perhaps x is a boundary point and y is a point of discontinuity, (that is, unless x is a right (left) hand limit of points in \mathcal{D}_i only and g is continuous from the left (right) at y).

Giving $g^{-1}(\text{int } \overline{\mathcal{D}_i}) \subseteq \overline{\mathcal{D}_i}$ and $g(\overline{\mathcal{D}_i}) \subseteq \overline{\mathcal{D}_i} \cup F_2$ where F_2 is at most an $m(m-1)$ point set. //

Lemma 3. For $i, j \in H_1$, $\overline{\mathcal{P}}_i = \overline{\mathcal{P}}_j$ or $\overline{\mathcal{P}}_i \cap \overline{\mathcal{P}}_j$ has empty interior.

Proof. Suppose $\overline{\mathcal{P}}_i \cap \overline{\mathcal{P}}_j$ has non empty interior.

Since $i, j \in H_1$ there are open intervals U_i and U_j in $\overline{\mathcal{P}}_i \cap \overline{\mathcal{P}}_j$ and integers M_i and M_j such that

- (i) $P_i \in g^{M_i} U_i$ and $\bigcup_{n=1}^{M_i-1} g^n U_i$ contains no discontinuity and
(ii) $P_j \in g^{M_j} U_j$ and $\bigcup_{n=1}^{M_j-1} g^n U_j$ contains no discontinuity.

By lemma 2, $P_i \in \text{int } \overline{\mathcal{P}}_j$ and $P_j \in \text{int } \overline{\mathcal{P}}_i$.

Again by lemma 2, $\{g^{-n} P_i\} \subseteq \overline{\mathcal{P}}_j$ for all n and $\{g^{-n} P_j\} \subseteq \overline{\mathcal{P}}_i$ for all n . Giving $\overline{\mathcal{P}}_i = \overline{\mathcal{P}}_j$. //

Lemma 4. $I = \bigcup_{i \in H_1} \overline{\mathcal{P}}_i$, or equivalently, for $j \in H_2$ $\overline{\mathcal{P}}_j = \bigcup_{i \in H_{1j}} \overline{\mathcal{P}}_i$,

where H_{1j} is a subset of H_1 ,

(for example in diagram 1 $\overline{\mathcal{P}}_2 = \overline{\mathcal{P}}_1 \cup \overline{\mathcal{P}}_3$).

Proof. Let $j \in H_2$ then there exist $j_1 \in H_1$ such that $g^{n_j} P_{j_1} = P_j$ and $\overline{\mathcal{P}}_{j_1} \subseteq \overline{\mathcal{P}}_j$.

By lemma 2 $P_j \in \overline{\mathcal{P}}_{j_1}$ and $\{g^{-n} P_j\} \subseteq \overline{\mathcal{P}}_{j_1}$ unless perhaps $\{g^{-n} P_j\}$ contains a point of discontinuity P_{j_2} , where $P_{j_2} \neq P_{j_1}$.

Then $g^{n'} P_{j_2} = P_j$ for some n' and $\overline{\mathcal{P}}_{j_1} \cup \overline{\mathcal{P}}_{j_2} \subseteq \overline{\mathcal{P}}_j$.

Again by lemma 2 $P_j \in \overline{\mathcal{P}}_{j_1} \cup \overline{\mathcal{P}}_{j_2}$ and $\{g^{-n} P_j\} \subseteq \overline{\mathcal{P}}_{j_1} \cup \overline{\mathcal{P}}_{j_2}$

unless $g^{n'} P_{j_3} = P_j$ for some $n' > 0$ and some discontinuity $P_{j_3} \neq P_{j_1}$,

$P_{j_3} \neq P_{j_2}$ in which case $\overline{\mathcal{P}}_j \supseteq \overline{\mathcal{P}}_{j_1} \cup \overline{\mathcal{P}}_{j_2} \cup \overline{\mathcal{P}}_{j_3}$.

Continuing thus, we finally have

$\overline{\mathcal{P}}_j = \overline{\mathcal{P}}_{j_1} \cup \dots \cup \overline{\mathcal{P}}_{j_k}$ for some $k \in \mathbb{Z}^+$.

Later we shall show $k = 2$. //

Lemma 5. For $i \in H_1 \cup H_2$, $\overline{\mathcal{D}}_i$ has no points of isolation, that is,

$$\overline{\text{int } \mathcal{D}_i} = \overline{\mathcal{D}}_i.$$

Proof. Let $i \in H_1$.

Suppose $\overline{\mathcal{D}}_i \setminus \overline{\text{int } \mathcal{D}_i} \neq \emptyset$.

Then there exists $y \in \{g^{-M}P_i\}$ with $y \in \overline{\mathcal{D}}_i \setminus \overline{\text{int } \mathcal{D}_i}$ for some $M \in \mathbb{Z}^+$.

By lemma 4 y is not an isolation point for some $\overline{\mathcal{D}}_j$ where $j \in H_1$ and $j \neq i$.

By lemma 2 $y \in \overline{\mathcal{D}}_j$ implies $g^n y \in \overline{\mathcal{D}}_j$ for all $n > 0$, unless $g^{n-1}y$ is a boundary point and a point of discontinuity.

But since $y \in \{g^{-M}P_i\}$ and $i \in H_1$, $y, \dots, g^{M-1}y$ are not points of discontinuity.

Therefore $g^M y \in \overline{\mathcal{D}}_j$, that is, $P_i \in \overline{\mathcal{D}}_j$.

However $P_i \in \text{int } \overline{\mathcal{D}}_i$ and by lemma 3 $\text{int } \overline{\mathcal{D}}_i \cap \text{int } \overline{\mathcal{D}}_j = \emptyset$ giving a contradiction.

Therefore $\overline{\mathcal{D}}_i = \overline{\text{int } \mathcal{D}_i}$. //

From lemmas 1 to 5 we see the sets $\{\overline{\mathcal{D}}_i\}_{i \in H_1}$ are good candidates for the sets I_1, \dots, I_N in Theorem 1, all but conditions (ii) and (iii) of this theorem being satisfied. We need to show that $\overline{\mathcal{D}}_i$, for $i \in H_1$, is a finite union of closed intervals.

Proof of Theorem 1. Let $i \in H_1$.

Let E_i be the largest (maximal) closed interval in $\overline{\mathcal{D}}_i$ containing P_i

and $E(i) = \bigcup_{P_k \in \text{int } \overline{\mathcal{D}}_i} E_k$, ($E(i)$ is the union of all maximal intervals

in $\overline{\mathcal{D}}_i$ containing any discontinuity in its interior).

Consider $g|_{E(i)}$. Let C be a maximal interval of this function,

that is, a maximal interval of continuity in $E(i)$.

Let $n_C \in \mathbb{Z}^+$ satisfy $(gC \cup \dots \cup g^{n_C-1}C) \cap E(i) = \emptyset$ and $g^{n_C}C \cap E(i) \neq \emptyset$,
 (n_C is the time of first return of C to $E(i)$).

Let $M_i = \max \{n_C : C \text{ is a maximal interval of } g|_{E(i)}\}$.

Each interval of $\bigcup_{n=1}^{M_i-1} g^n E(i)$ is contained in a maximal interval of $\overline{\mathcal{P}}_i$.

Let $\mathcal{E}(i)$ be the union of all these maximal intervals, that is,

$$\mathcal{E}(i) \supseteq \bigcup_{n=1}^{M_i-1} g^n E(i) \text{ and } \mathcal{E}(i) \text{ is a finite union of intervals in } \overline{\mathcal{P}}_i.$$

By construction $g\mathcal{E}(i) \subseteq \mathcal{E}(i)$.

We show that $\mathcal{E}(i) = \overline{\mathcal{P}}_i$ by showing that $\mathcal{E} = \bigcup_{i \in H_1} \mathcal{E}(i)$ is the full interval I .

By lemma 4 we know that $\bigcup_{n=0}^{\infty} g^{-n} \mathcal{E} = I$.

Note also, for $j \in H_2$ we have $P_j \in \mathcal{E}$. This follows from lemma 4, since $\overline{\mathcal{P}}_j = \overline{\mathcal{P}}_{j_1} \cup \dots \cup \overline{\mathcal{P}}_{j_k}$, $j_1, j_2, \dots, j_k \in H_1$ where $g^{n_i} P_{j_i} = P_j$ for some $n_i \in \mathbb{Z}^+$ and $i=1, \dots, k$. Therefore $P_j \in E(j_i)$ for $i=1, \dots, k$.

Suppose $\mathcal{E} \neq I$. We consider separately the following possibilities

- (i) $\overline{\mathcal{E}^c}$ contains no points of discontinuity,
- (ii) Each discontinuity in $\overline{\mathcal{E}^c}$ is an end point of two intervals in some $\overline{\mathcal{P}}_j$ and $\overline{\mathcal{P}}_i$,
- (iii) $\overline{\mathcal{E}^c}$ contains a discontinuity P_k which is not the end point to two intervals in some $\overline{\mathcal{P}}_j$ and $\overline{\mathcal{P}}_i$, that is, P_k is a limit point of intervals of some $\overline{\mathcal{P}}_j$ but is contained in no interval of $\overline{\mathcal{P}}_j$.

We show these three exhaustive possibilities all lead to contradictions, showing that $\mathcal{E} = I$.

(1) $\overline{\mathcal{E}^c}$ is a finite union of closed intervals containing no points of discontinuity.

$$g^{-1}\overline{\mathcal{E}^c} \subset \overline{\mathcal{E}^c} \quad \text{and} \quad g\overline{\mathcal{E}^c} \supset \overline{\mathcal{E}^c}.$$

Since g is continuous on the boundary of $\mathcal{E}(\partial\mathcal{E})$,

$g\partial\mathcal{E} = \partial\mathcal{E}$ and $\partial\mathcal{E}$ consists of periodic points.

Let M be the l.c.m of the periods of points on $\partial\mathcal{E}$.

g^M fixes $\partial\mathcal{E}$.

Since $g\mathcal{E} \subseteq \mathcal{E}$ and g is onto, g maps maximal intervals of $\overline{\mathcal{E}^c}$ onto maximal intervals of $\overline{\mathcal{E}^c}$.

Let $F = [c, d]$ be a maximal interval of $\overline{\mathcal{E}^c}$ where $c \in \partial\mathcal{E}(i)$ and $d \in \partial\mathcal{E}(j)$ for some $i, j \in H_1$.

Now g^M is continuous at, and fixes, c and d ; and g^M maps F onto itself.

See diagram 2.

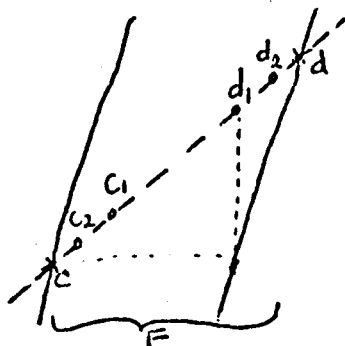


Diagram 2.

$$\begin{array}{c} g^M \\ | \\ F \end{array}$$

Consequently we have sequences (c_n) and (d_n) such that $c_n \in \{g^{-nM}d\}$, $d_n \in \{g^{-nM}c\}$ and $c_n \downarrow c$ and $d_n \uparrow d$.

By lemma 2 $c_n \in \overline{\mathcal{P}}_j$ and $d_n \in \overline{\mathcal{P}}_i$ for all n .

Thus $c \in \overline{\mathcal{P}}_j$ and $d \in \overline{\mathcal{P}}_i$ giving $i = j$.

Therefore F lies between two intervals of the same set $\mathcal{E}(i)$. (3.2)

By the cyclic nature of intervals in $\mathcal{E}(i)$ and the (3.3)

continuity of g on $\partial\mathcal{E}(i)$, there are intervals of $\overline{\mathcal{E}^c}$

on either side of $\mathcal{E}(i)$ and hence on either side of each interval in $\mathcal{E}(i)$.

But statements (3.2) and (3.3) are incompatible.

(ii) $\overline{\mathcal{E}}^c$ contains a discontinuity P_k .

By the note on page 3.8, $P_k \in H_0$ and by the definition of \mathcal{E} , P_k is a boundary point of at least two sets $\overline{\mathcal{P}}_j, \overline{\mathcal{P}}_i$ for $j, i \in H_1$.

In this case we assume P_k is the end point of intervals of $\overline{\mathcal{P}}_i$ and $\overline{\mathcal{P}}_j$.

Let J be an interval in $\overline{\mathcal{E}}^c$ with $P_k \in J$.

There exist points $a_i \in \overline{\mathcal{P}}_i$ and $b_j \in \overline{\mathcal{P}}_j$ such that

$$[a_i, P_k] \subseteq \overline{\mathcal{P}}_i \quad \text{and} \quad [P_k, b_j] \subseteq \overline{\mathcal{P}}_j.$$

$J \cap [a_i, b_j]$ is a closed interval, in fact $[a_i, b_j] \subseteq J$.

There exists a positive integer M'_k such that $g^{M'_k}[a_i, b_j] \subseteq \mathcal{E}$.

Thus $g^{M'_k} P_k \in \text{int } \mathcal{E}$.

Let $M = \max_{P_k \in \overline{\mathcal{E}}^c} (M'_k)$.

Then $g^M P_k \in \text{int } \mathcal{E}$, for all $P_k \in \overline{\mathcal{E}}^c$.

Consider $g^{-M}(\overline{\mathcal{E}}^c)$.

This does not contain any points of discontinuity and

$$g^{-1}(g^{-M}(\overline{\mathcal{E}}^c)) \subseteq (g^{-M}\overline{\mathcal{E}}^c).$$

Also $g(g^{-M}\mathcal{E}) \subseteq (g^{-M}\mathcal{E})$ and $g^{-M}\mathcal{E}$ is a finite union of intervals.

Therefore by similar reasoning to case (i) we can deduce

that $g^{-M}\overline{\mathcal{E}}^c = \emptyset$. But g is onto, so this implies $\overline{\mathcal{E}}^c = \emptyset$.

(iii) $\overline{\mathcal{E}}^c$ contains a discontinuity P_k which is the limit point of some set $\overline{\mathcal{P}}_j$ but P_k is not contained in any interval of $\overline{\mathcal{P}}_j$, $j \in H_1$.

Let $S = \{z \in I : z \text{ is a limit point of some } \overline{\mathcal{P}}_j \text{ but } z \text{ is not contained in any interval of } \overline{\mathcal{P}}_j, j \in H_1\}$.

By assumption $S \neq \emptyset$ for $P_k \in S$.

Let z be an isolated point in S and let

$$H_z = \{i \in H_1 : z \in \overline{\mathcal{P}}_i\}.$$

Without loss in generality we may assume that z is approximated from above by intervals in $\bigcup_{j \in H_z} \overline{\mathcal{P}}_j$.

Let $N_\delta(z) = [z, z+\delta]$ be a neighbourhood of z such that $(z, z+\delta]$ does not contain any discontinuity and $N_\delta(z) \cap S = \emptyset$.

$N_\delta(z)$ is a countable union of intervals in $\bigcup_{j \in H_z} \overline{\mathcal{P}}_j$.

Choose $\delta_1 > 0$ such that $[z+\delta_1, z+\delta]$ contains at least three intervals of each set $\overline{\mathcal{P}}_i$, $i \in H_z$. (3.4)

$[z+\delta_1, z+\delta]$ is a finite union of sets in $\bigcup_{i \in H_z} \overline{\mathcal{P}}_i$ since

$$N_\delta(z) \cap S = \emptyset.$$

Therefore there exists a natural number M such that

$$g^M [z+\delta_1, z+\delta] \subset \mathcal{E}.$$

From which we deduce, using condition (3.4), that each interval of $\mathcal{E}(i)$, $i \in H_z$, is bounded on either side by intervals in $\bigcup_{j \in H_z} \mathcal{E}(j)$.

Therefore $\bigcup_{i \in H_k} \mathcal{E}(i)$ is an interval and has no boundary, that is, $\bigcup_{i \in H_z} \mathcal{E}(i) = I$, which contradicts the initial assumption. //

Corollary 1. f -expansions in \mathcal{C} are irreducible.

Proof. Let $g \in \mathcal{C}$ be an f -expansion.

Let P_i be a discontinuity of g then $P_i \in \text{int } \overline{\mathcal{P}}_i$.

Let $U_\varepsilon(P_i)$ be an ε -neighbourhood of P_i contained in $\overline{\mathcal{P}}_i$.

Then $g(U_\varepsilon(P_i))$ is the union of two intervals of 0 and 1.

Thus $0 \in \text{int } \overline{\mathcal{P}}_i$ and $1 \in \text{int } \overline{\mathcal{P}}_i$. This is true for all i , so by

Theorem 1 $\overline{\mathcal{P}}_i = I$. //

§3. Kneading Sequences and Related Power Series.

Throughout we consider a fixed $g \in \mathcal{C}$ with discontinuities

P_1, \dots, P_{m-1} .

Let $\mathcal{D} = \bigcup_{N=0}^{\infty} g^{-N} \bigcup_{n=0}^{\infty} g^n \{P_1, \dots, P_{m-1}\}$ and $I_1 = I \setminus \mathcal{D}$.

If $x \in I_1$ let $\theta_0(x) = i$ if $x \in (P_i, P_{i+1})$, $i=0, \dots, m-1$, $P_0=0$, $P_m=1$

$\theta_0(0) = 0$ if $0 \in I_1$,

$\theta_0(1) = m-1$ if $1 \in I_1$,

and let $\theta_n(x) = \theta_0(g^n(x))$ for $n \in \mathbb{N}$.

Definition. The formal power series $\theta(x) = \sum_{n=0}^{\infty} \theta_n(x) t^n$ is called the Kneading series of x , $x \in I_1$.

Let $\mathbb{Z}[t]$ be the space of formal power series with integer coefficients.

We endow $\mathbb{Z}[t]$ with the lexicographical ordering

$\sum_{n=0}^{\infty} \alpha_n t^n < \sum_{n=0}^{\infty} \alpha'_n t^n$ if the first non zero coefficient of $\sum_{n=0}^{\infty} (\alpha'_n - \alpha_n) t^n$ is positive.

We also give $\mathbb{Z}[t]$ the topology induced by the metric

$$\rho\left(\sum_{n=0}^{\infty} \alpha_n t^n, \sum_{n=0}^{\infty} \alpha'_n t^n\right) = \sum_{n=0}^{\infty} |\alpha_n - \alpha'_n| 2^{-n}.$$

Lemma 2. θ is a strictly monotone increasing continuous function on I_1 .

Proof. Suppose $x <_1 y$, $x, y \in I_1$.

Let n_0 be the smallest natural number such that

$(g^{n_0}x, g^{n_0}y) \cap \{P_1, \dots, P_{m-1}\} \neq \emptyset$. n_0 exists as $\{P_1, \dots, P_{m-1}\}$

separate points.

g^{n_0} is monotone increasing on its maximal intervals, so

$\theta_j(x) = \theta_j(y)$ for $0 \leq j < n_0$ and $\theta_{n_0}(x) < \theta_{n_0}(y)$.

Thus θ is strictly monotone increasing.

Let $x_1 \in I_1$. Given $\varepsilon > 0$, choose N such that $m2^{-N} < \varepsilon$.

Now choose $\delta > 0$ such that $(x_1 - \delta, x_1 + \delta) \cap \bigcup_0^N g^{-j} \{P_1, \dots, P_{m-1}\} = \emptyset$.

If $y \in I_1 \cap (x_1 - \delta, x_1 + \delta)$ then

$$|\theta(y) - \theta(x_1)| < 1/2^N |\theta(g^N y) - \theta(g^N x_1)| < m2^{-N} < \varepsilon.$$

Therefore θ is continuous on I_1 . //

The following limits exist for all $x \in I$,

$$\theta^+(x) = \lim_{\substack{y \downarrow x \\ y \in I_1}} \theta(y) \quad \text{and} \quad \theta^-(x) = \lim_{\substack{y \uparrow x \\ y \in I_1}} \theta(y).$$

For $x \in I_1$, $\theta^-(x) = \theta(x) = \theta^+(x)$.

The functions θ^- and θ^+ are defined on I . In particular,

$\theta^-(P_i)$ and $\theta^+(P_i)$ for $i=1, \dots, m-1$ are defined.

Definition. Let $\psi_i(t)$ be the formal power series defined by

$$\psi_i(t) = \theta^+(P_i) - \theta^-(P_i), \quad (1)$$

then ψ_i is called the i^{th} kneading invariant of g , $i=1, \dots, m-1$.

If $g^M(x) = P_i$, for some i , where M is the smallest positive integer satisfying this, then $\theta^+(x) - \theta^-(x) = t^M \psi_i(t)$. (2)

Definitions. Let J be an interval of I .

(i) Let $\gamma_{\text{in}}(J)$ be the number of points in

$$\{ \{g^{-n}P_i\} \setminus \{g^{-n+1}P_i\} \} \cap \text{int} J \quad \text{and denote } \gamma_{\text{in}}(I) \text{ by } \gamma_{\text{in}}.$$

Let $\beta_i = \limsup_n \sqrt[n]{\gamma_{\text{in}}}$ and $r_i = 1/\beta_i$ for $i=1, \dots, m-1$.

(ii) Let $l_n(J)$ be the number of maximal intervals of $g^n|_J$

and denote $l_n(I)$ by l_n .

$\beta = \limsup_n \sqrt[n]{l_n}$ and let $r = 1/\beta$.

(iii) Let $\gamma_i(J, t)$ and $L(J, t)$ be the formal power series defined by

$$\gamma_i(J, t) = \sum_{n=0}^{\infty} \gamma_{in}(J) t^n \quad \text{for } i=1, \dots, m-1 \quad \text{and}$$

$$L(J, t) = \sum_{n=0}^{\infty} l_n(J) t^n ; \quad \text{and}$$

denote $\gamma_i(I, t)$ and $L(I, t)$ by $\gamma_i(t)$ and $L(t)$ respectively.

Notice that
$$l_n = 1 + \sum_{j=0}^n \sum_{i=1}^{m-1} \gamma_{ij} \quad (3)$$

so we have as a consequence

$$L(t) = (1+t)^{-1} (1-t)^{-1} (\gamma_1(t) + \dots + \gamma_{m-1}(t)) . \quad (4)$$

Lemma 3. Given any interval $J = [a, b]$ (or $J = (a, b)$) in I ,

$$\begin{aligned} \theta^-(b) + \theta^+(a) &= \sum_{i=1}^{m-1} \sum_{n=0}^{\infty} \gamma_{in}(J) v_i t^n \\ &= \sum_{i=1}^{m-1} v_i \sum_{n=0}^{\infty} \gamma_{in}(J) t^n . \end{aligned}$$

Proof. The maps $x \rightarrow \theta^-(x) \bmod t^{n+1}$ and $x \rightarrow \theta^+(x) \bmod t^{n+1}$ are step functions with the same set of discontinuities, and they are equal on their points of continuity.

Let z be a point of discontinuity of $\theta^-(x) \bmod t^{n+1}$, that is, $g^s z = P_i$ for some $i=1, \dots, m-1$ and $0 \leq s \leq n$.

Let s be the smallest integer satisfying this, then the size of the discontinuity of $\theta^-(x) \bmod t^{n+1}$ at z is

$$\theta^+(z) - \theta^-(z) \bmod t^{n+1} = t^s v_i(t) \bmod t^{n+1}, \quad \text{by equation (2).}$$

There are $\gamma_{is}(J)$ such discontinuities in $\text{int} J$ for each $i=1, \dots, m-1$.

By the monotone increasing property of θ^+ and θ^- (inherited from θ), we have

$$\theta^-(b) - \theta^+(a) \bmod t^{n+1} = \sum_{i=1}^{m-1} \sum_{j=0}^n t^j \gamma_{ij}(J) v_i \bmod t^{n+1}.$$

let $n \rightarrow \infty$, then

$$\theta^-(b) - \theta^+(a) = \sum_{i=1}^{m-1} v_i \sum_{n=0}^{\infty} \gamma_{in}(J) t^n. //$$

In particular, we have

$$\underline{\theta^-(1) - \theta^+(0) = v_1 \sum_{n=0}^{\infty} \gamma_{1n} t^n + \dots + v_{m-1} \sum_{n=0}^{\infty} \gamma_{m-1n} t^n.} \quad (5)$$

§4. Linear Models.

g is fixed as in §3. Consider now the formal power series θ^- , θ^+ , L , ψ_i and γ_i $i=1, \dots, m-1$, as power series in the complex variable t .

The radii of convergence of θ^- , θ^+ and $\psi_1, \dots, \psi_{m-1}$ are 1 and the radii of convergence for γ_i and L are r_i and r respectively, for $i=1, \dots, m-1$.

As $g \in \mathcal{C}$ we have $r < 1$ so $r_i < 1$ for some $i=1, \dots, m-1$ and $r = \min_i r_i$ by equation (4).

We see from equation (5) that if $r_i < 1$, ψ_i has a zero z_i on $|t| = r_i$.

But for $N \geq 0$, $\sum_{n=0}^N \gamma_{in} r_i^n = \sum_{n=0}^N \gamma_{in} |z_i|^n \geq \left| \sum_{n=0}^N \gamma_{in} z_i^n \right|$,

giving $z_i = r_i$.

It also follows from (5) that $\psi_i(t)\gamma_i(t)$ is analytic on $|t| < 1$, so if $r_i < 1$, $\gamma_i(t)$ has a removeable singularity at r_i and extends to a meromorphic function of $|t| \leq 1$.

We shall need to consider the following subsets of $\{1, 2, \dots, m-1\}$.

$$F_1 = \{i: i \in H_1 \text{ and } \overline{\psi}_i \neq \overline{\psi}_j \text{ for } 1 \leq j < i\}$$

$$G = \{i: r_i < 1 \text{ for } i \in H_1\}.$$

The above information allows us to define certain measures on I .

Definition. Given an interval $J \subset I$ and $i \in G$ let

$$\Lambda_i(J) = \lim_{t \uparrow r_i} \gamma_i(J, t) / \gamma_i(t)$$

$$\Lambda(J) = \lim_{t \uparrow r} L(J, t) / L(t).$$

Since $0 \leq \gamma_i(J, t) \leq \gamma_i(t)$ for $0 \leq t < r_i$ and

$0 \leq L(J, t) \leq L(t)$ for $0 \leq t < r$, the above ratios remain bounded as $t \uparrow r_i$ and $t \uparrow r$ respectively, and the apparent singularities at r_i and r are removable, and therefore the above definition makes sense.

$$0 \leq \Lambda_i(J) \leq 1 \quad \text{and} \quad 0 \leq \Lambda(J) \leq 1.$$

We can regard $\Lambda(J)$ as the probability that a randomly chosen point of I will lie in the interval J .

Lemma 4. Λ_i and Λ are non-atomic measures, that is,

$$(i) \quad \Lambda_i(J_1 \cup J_2) = \Lambda_i(J_1) + \Lambda_i(J_2)$$

$\Lambda(J_1 \cup J_2) = \Lambda(J_1) + \Lambda(J_2)$ whenever J_1 and J_2 are intervals with disjoint interiors.

(ii) $\Lambda_i(J)$ and $\Lambda(J)$ depend continuously on the end points of J .

and in addition

(iii) if $J \subset (P_j, P_{j+1})$, for some $j=0, \dots, m-1$, then

$$\Lambda_i(gJ) = \beta_i \Lambda_i(J) \quad \text{and} \quad \Lambda(gJ) = \beta \Lambda(J).$$

Proof. We prove the lemma for Λ only, as similar reasoning applies to Λ_i also.

(i) $\Lambda(J_1) + \Lambda(J_2)$ differs from $\Lambda(J_1 \cup J_2)$ by at most 1.

$$\text{So } |L(J_1, t) + L(J_2, t) - L(J_1 \cup J_2, t)| \leq \sum_{n=0}^{\infty} t^n \quad \text{and}$$

this remains bounded as $t \uparrow r$.

$$\text{Thus } \Lambda(J_1) + \Lambda(J_2) - \Lambda(J_1 \cup J_2) = 0.$$

(iii) $\Lambda_{n+1}(J) = \Lambda_n(gJ)$ for J in a maximal interval of g ,

$$\text{therefore } L(J, t) = 1 + tL(gJ, t) \quad \text{and} \quad L(J) = r\Lambda(gJ).$$

(ii) If J is small enough to be contained in a maximal interval of g^n , then by (iii)

$\Lambda(J) = r^n \Lambda(g^n J) \leq r^n$, which converges to zero as n increases. //

Definition. $\lambda(x) = \Lambda([0, x])$ and $\lambda_i(x) = \Lambda_i([0, x])$ for $i \in G$.

λ and λ_i are continuous and monotone increasing maps of I , by lemma 4. We show that they commute with uniformly piecewise linear maps of I .

Definition. T is a piecewise linear map of I if there are points q_0, \dots, q_m such that

$$(i) \quad 0 = q_0 < q_1 < \dots < q_m = 1 \quad \text{and}$$

$$(ii) \quad T \Big|_{(q_i, q_{i+1})}(x) = b_i x + a_i, \text{ where } 0 \leq a_i < 1, \quad b_i q_{i+1} + a_i \leq 1$$

$$\text{and } 0 \leq b_i q_i + a_i.$$

T is uniformly piecewise linear if $b_i = b$ for $i=0, \dots, m-1$.

Theorem 2. For each map λ, λ_j ($j \in G$) there exist unique uniformly piecewise linear maps T, T_j such that

$$\lambda(g(x)) = T(\lambda(x)) \quad \text{and} \quad \lambda_j(g(x)) = T_j(\lambda_j(x)) \quad \text{where}$$

T and T_j have gradients β and β_j respectively.

Proof. Note, λ and λ_j are continuous but not in general one-one.

For a well defined map T (T_j) to exist we must show that if

$$\lambda(x) = \lambda(y) \quad (\lambda_j(x) = \lambda_j(y)) \quad \text{then} \quad \lambda(g(x)) = \lambda(g(y)) \quad (\lambda_j(g(x)) = \lambda_j(g(y))).$$

Suppose $y > x$ and $\lambda(x) = \lambda(y)$ ($\lambda_j(x) = \lambda_j(y)$) then

$$\Lambda([x, y]) = 0 \quad (\Lambda_j([x, y]) = 0) \quad \text{and by lemma 4}$$

$$\Lambda([g(x), g(y)]) = 0 \quad (\Lambda_j([g(x), g(y)]) = 0) \quad \text{thus}$$

$$\lambda(g(x)) = \lambda(g(y)) \quad (\lambda_j(g(x)) = \lambda_j(g(y))).$$

Define $T(x) = \beta x + \lambda(b_k) - \beta \lambda(p_k)$ for $x \in (\lambda_{p_k}, \lambda_{p_{k+1}})$ where

$$b_k = \lim_{y \downarrow p_k} g(y), \quad k=1, \dots, m-1.$$

T is a uniformly piecewise linear map on $I \setminus \{\lambda_{P_1}, \dots, \lambda_{P_{m-1}}\}$.

Let $x \in (P_k, P_{k+1})$ then $T\lambda(x) = \beta\lambda(x) + \lambda(b_k) - \beta\lambda(P_k)$, $k=1, \dots, m-1$,

$$\begin{aligned} \text{and } \lambda_{g(x)} &= \Lambda([0, g(x)]) = \Lambda([0, b_k]) + \Lambda([b_k, g(x)]) \\ &= \lambda_{b_k} + \beta\Lambda([P_k, x]) \\ &= \lambda_{b_k} + \beta\Lambda([0, x]) - \beta\Lambda([0, P_k]) \\ &= \lambda_{b_k} + \beta\lambda(x) - \beta\lambda(P_k) \\ &= T\lambda(x). \end{aligned}$$

Define T at λ_{P_k} by $T\lambda_{P_k} = \lambda_{g(P_k)}$ for $k=1, \dots, m-1$, then

T is uniformly piecewise linear on I .

Define $T_j(x) = \beta_j(x) + \lambda_j(b_k) - \beta_j\lambda_j(P_k)$ for $x \in \lambda_j(P_k, P_{k+1})$

where $b_k = \lim_{y \downarrow P_k} g(P_k)$ and

$$T_j(\lambda_{P_k}) = \lambda_{g(P_k)} \quad \text{for } k=1, \dots, m-1.$$

Then as above $T_j(\lambda_j(x)) = \lambda_j(g(x))$.

Uniqueness of these linear maps is forced by the commutative condition. //

Note, from equation (4) $\lambda(x) = \sum_{j \in G} c_j \lambda_j(x)$ where $c_j = 0$ if $r_j > r$

and $c_j = (1-r)^{-1} \lim_{t \rightarrow r} \gamma_i(t)/L(t)$ if $r_j = r$.

Corollary 1. Suppose g is indecomposable, that is $\overline{\theta}_i = I$ for all $i \in H_1$, then λ is a homeomorphism, as are λ_i for $i \in H_1$.

Proof. Suppose λ is not a homeomorphism then there is a closed interval J on which λ is constant, therefore $\Lambda(J) = 0$.

Let N be the smallest non-negative integer such that for some $i \in H_1$, $g^N(\text{int } J) \cap \{P_i\} \neq \emptyset$ and $J \cup \dots \cup g^{N-1}(J)$ does not contain any discontinuity of g .

Then $\Lambda(J) = \Lambda(gJ) = \dots = \Lambda(g^N(J)) = 0$ by lemma 4, giving an open interval containing P_i with Λ measure zero.

Let K be the union of all intervals with zero Λ -measure.

By lemma 4, K is essentially g -invariant and contains P_i in its interior.

Therefore $K = \overline{P}_i = I$ which is impossible.

Similar reasoning gives the result for $\lambda_i, i \in H_1$, also. //

Comment 1. Corollary 1 generalizes the result in [P2] where it was required that g be strongly transitive (g is strongly transitive if for all open intervals $U \subset I$ there exists N such that $g^N U = I$). Strong transitivity implies, but is not implied by, the properties $\beta > 1$, $\overline{P}_i = I$ for all $i \in H_1$ and $\{P_1, \dots, P_{m-1}\}$ separating points.

We now determine conditions on $g \in \mathcal{C}$ under which g is topologically conjugate to a piecewise linear map.

Suppose $G = H_1$.

By Theorem 1 I is decomposable into irreducible sets $\{\overline{P}_i\}_{i \in F_1}$ where \overline{P}_i is a finite union of closed intervals and essentially invariant under g .

So by restricting g to each \overline{P}_i we can use Corollary 1 of Theorem 2 to get $g|_{\overline{P}_i}$ topologically conjugate to a uniformly piecewise linear transformation T_i on \overline{P}_i . Using this we can construct a conjugacy of g with a piecewise linear map.

Theorem 3. Let $g \in \mathcal{C}$ such that $G = H_1$ (that is, $r_i < 1$ for all $i \in H_1$) then g is topologically conjugate to a piecewise linear map T of the unit interval, where T has essentially invariant sets R_i such that the slope of T restricted to R_i is $r_i^{-1} = \beta_i$.

Proof. Let $\overline{\lambda}(x) = \sum_{j \in F_1} \delta_j \lambda_j(x)$ where δ_j are any normalizing factors such that $\sum_{j \in F_1} \delta_j = 1$ and $\delta_j \neq 0$.

(the most obvious choice for δ_j is the Lebesgue length of $\overline{\mathcal{P}}_j$).

$\overline{\lambda}$ is certainly a continuous homeomorphism since λ_j is a homeomorphism when restricted to $\overline{\mathcal{P}}_j$ and constant on $\overline{\mathcal{P}}_i \neq \overline{\mathcal{P}}_j$.

Let $T_i(x) = \lambda_i g \lambda_i^{-1}(x)$ ^{for $x \in \overline{\mathcal{P}}_i$} then T_i is uniformly linear ^{on $\lambda_i \overline{\mathcal{P}}_i$} by Theorem 2.

Let $T = \overline{\lambda} g \overline{\lambda}^{-1}$ then we must show T is piecewise linear.

T certainly has discontinuities $\{\lambda_{P_i}\}_{i=1}^{m-1}$.

Let $x \in \overline{\lambda} \overline{\mathcal{P}}_i$ then $T(x) = \overline{\lambda} g \overline{\lambda}^{-1}(x)$

$$\begin{aligned} &= \sum_{j \in F_1} \delta_j \lambda_j g \lambda_j^{-1}(x) \\ &= \delta_i T_i(x) + \sum_{\substack{j \in F_1 \\ j \neq i}} \delta_j T_j(x) \end{aligned}$$

Now $\sum_{\substack{j \in F_1 \\ i \neq j}} \delta_j T_j(x)$ is constant on $\overline{\lambda} \overline{\mathcal{P}}_i$ thus $T|_{\overline{\lambda} \overline{\mathcal{P}}_i}$ is uniformly

piecewise linear, with gradient β_i .

Let $R_i = \overline{\lambda} \overline{\mathcal{P}}_i$ then the result follows. //

Note, the homeomorphism $\overline{\lambda}$ is not unique, apart from trivially depending on the constants δ_i , $\overline{\lambda}$ also depends on F_1 . If we choose a different indexing set to represent the distinct sets $\overline{\mathcal{P}}_i$ we may get a different homeomorphism. Thus $\overline{\lambda}$ is unique (up to constants δ_i) if each set $\overline{\mathcal{P}}_i$ contains only one discontinuity.

Corollary 2. If $g \in \mathcal{C}$, $G = H_1$ and $r_i = r$ for all $i \in G$ then g is topologically conjugate to a uniformly piecewise linear map of I .

Comment 2. Suppose we consider the class of piecewise monotone increasing transformations of the unit interval with growth number greater than 1 (but whose points of discontinuity do not necessarily separate points). Then by similar analysis to that

in §§3 and 4 we can show that these transformations are semiconjugate to certain uniformly piecewise linear maps.

Comment 3. We have restricted our attention here to piecewise monotone increasing transformations, however in [P2] only piecewise monotonicity was required on the maximal intervals of continuity. By introducing an orientation coordinate also to each maximal interval and a different ordering on $\mathbb{Z}[t]$, it seems likely that Theorem 2 would hold for this larger class also.

Comment 4. Instead of using the above complex analytic approach, it should be possible to generalize the methods in [P2]. In [P2] the symbolic representation (\sum_g, S) of g is considered, for g strongly transitive, (where S is the shift on $\sum_g = \prod_0^\infty \{0, \dots, n-1\}$). This representation (\sum_g, S) is approximated by irreducible subshifts of finite type containing \sum_g , for which conjugacies are known to exist. These subshifts of finite type arise naturally as experimental approximations to the process (\sum_g, S) over finite time and their growth numbers approximate that of (\sum_g, S) .

There is an obvious correspondence between the meaning sequences of g and the points of its symbolic representation and in fact the conjugacy constructed in Theorem 2 for strongly transitive transformations is, in essence, the same as that constructed in [P2].

To prove Theorems 2 and 3 by these methods we need a more general class of subshifts, similar to, and including the subshifts of finite type, whose dynamics are 'simple' to understand and with which we can approximate (\sum_g, S) . This approach would have the advantage of giving more information concerning the dynamics and structure of the maps in \mathcal{C} and could lead to certain density and category results for these 'generalized subshifts of finite type'.

§5. Measure Theoretic Conclusions.

Using the semi-conjugacy constructed in Theorem 2 we can carry over invariant measures of the uniformly linear transformations to give invariant measures for transformations in \mathcal{C} . Li and Yorke have shown that any piecewise linear map has an invariant measure absolutely continuous with respect to Lebesgue measure 1 . Thus any map $g \in \mathcal{C}$ has an invariant measure.

Suppose $g \in \mathcal{C}$, where g is piecewise differentiable and $1/g'$ is of bounded variation, then g preserves an ergodic measure μ absolutely continuous with respect to 1 [Wo]. For these transformations one might ask whether the conjugacies or semi-conjugacies constructed in Theorems 2 and 3 are absolutely continuous maps. However the following simple example, shown to me by W. Parry, demonstrates that this is not true in general.

Let f be the map defined by $f(x) = 3x$ for $x < \frac{1}{3}$
and $f(x) = 3x/2 - \frac{1}{2}$ for $x \geq \frac{1}{3}$.

f preserves Lebesgue measure 1 and this is the only smooth measure preserved by f [Li.Y]. The entropy of f with respect to 1 is $h_1(f) = \frac{1}{3} \log 3 + 2/3 \log 3/2$. Now $f \in \mathcal{C}$ and f is conjugate to a uniformly piecewise linear map $T(x) = 2x \bmod 1$. T also preserves 1 and this is the maximal measure of T [P2], with entropy $h_1(T) = \log 2$. If λ is the conjugacy of Theorem 2 then 1λ is the invariant maximal measure of f and $h_{1\lambda}(f) = \log 2$ thus $1\lambda \neq 1$ and λ is not an absolutely continuous map.

Suppose $g \in \mathcal{C}$ is an f -expansion then by the corollaries to Theorems 2 and 3 g is conjugate to a linear mod 1 map and so has a unique maximal measure with entropy $\log \beta$. The mixing properties of f with respect to this maximal measure are determined or referred to in Section 2.

SECTION 4.

Abbreviations.

Listed here are standard abbreviations not defined in the text.

\mathbb{C}	The complex numbers.
\mathbb{R}	The real numbers
\mathbb{Z}	The integers , $\dots -2, -1, 0, 1, 2, \dots$.
\mathbb{Z}^+	The positive integers , $1, 2, 3, \dots$.
\mathbb{N}	The natural numbers , $0, 1, 2, \dots$.
$C(X)$	The space of all continuous functions of the space X .
$C^2(X)$	The space of all continuously twice differentiable functions of X .
$L^1(X)$	The space of integrable functions of X .
$L^2(X), L^2$	The Hilbert space of square integrable functions of X .
χ_A	The characteristic function of $A \subseteq X$.
a.e.	almost everywhere.
a.a	almost all.
$\mu \ll l$	μ is absolutely continuous with respect to l .
$\text{spt } \mu$	The support of the measure μ .
$\text{int } J$	The interior of the set J .
\bar{J}	The closure of the set J .
$\partial(J)$	The boundary of the set J .

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